CS206 Homework #1

Max marks: 70 **Due Feb 16, 2006**

- *Be brief, complete and stick to what has been asked.*
- ***Do not copy solutions from others.***

1. *[10 + 10 + 10 marks]* Use natural deduction to prove the following sequents. You must use only the basic rules of natural deduction (no derived rules, including LEM, are allowed). Your proof using the basic rules must not exceed the number of steps mentioned alongside each sequent. The number of steps includes the statement of the premises. You must also annotate each step of your proof with the basic rule applied at that step.

   (a) $x_1 \rightarrow x_2 \vee x_3 \vee x_4,\ x_2 \rightarrow \neg x_1 \vee \neg x_4,\ x_3 \vee x_4 \rightarrow x_2 \ \vdash\ x_1 \rightarrow \neg x_4$ *[within 27 basic steps]*

   (b) $x_1 \vee x_2,\ \neg(x_1 \wedge x_2)\ \vdash\ (x_1 \wedge \neg x_2) \vee (x_2 \wedge \neg x_1)$ *[within 19 basic steps]*

   (c) $x_1 \rightarrow x_2 \vee x_3,\ x_2 \rightarrow \neg x_1 \vee \neg x_4,\ x_3 \rightarrow \neg x_1 \vee \neg x_5,\ x_4 \rightarrow x_1 \wedge x_5,\ x_5 \rightarrow x_1 \wedge x_4,\ x_1 \rightarrow x_4 \vee x_5\ \vdash$ $x_1 \rightarrow \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5$ *[within 40 basic steps]*

2. *[20 marks]* An undirected graph G can be conveniently represented by a symmetric adjacency matrix, A_G. If G has n nodes, A_G is an $n \times n$ matrix, such that $A_G[i][j] = 1$ if there is an edge between nodes i and j, and $A_G[i][j] = 0$ otherwise. Given such an adjacency matrix A_G and an integer $k > 0$, we wish to determine if it is possible to colour the nodes of G using at most k distinct colours, such that every pair of nodes that is connected by an edge are coloured differently. In other words, both ends of an edge cannot be coloured the same.

   You are required to describe a technique for reducing an arbitrary instance of the above graph colouring problem to an instance of propositional satisfiability. Thus, you must describe a systematic and mechanizable procedure that

   - takes as inputs an $n \times n$ symmetric matrix A_G describing an undirected graph G, and an integer $k > 0$, and
   - generates a propositional logic formula $\phi_{G,k}$,

   such that determining the (un)satisfiability of $\phi_{G,k}$ would tell us whether G can be coloured with at most k colours satisfying the constraint above. You must also ensure that the size of $\phi_{G,k}$ is no larger than a polynomial in n (no. of vertices in G) and k (no. of colours).

   You must clearly describe your procedure in pseudo-code format, and show the formula that will be generated by your procedure for the graph G represented by the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

3. *[10 + 10 marks]* A student wishes to find a satisfying assignment for the propositional logic formula $\phi = \phi_1 \wedge \phi_2$, where ϕ_1 is a Horn formula and ϕ_2 is a CNF formula that is not a Horn formula. Formulae ϕ_1 and ϕ_2 have some common propositional variables between them. The student proposes to proceed in two different ways to solve this problem:

(a) A satisfying assignment for ϕ_1 is obtained using the method for satisfiability checking of Horn formulae discussed in class (i.e., determine which propositional variables must be set to True because of implications, and then set all the remaining variables to False). Formula ϕ_2 is then simplified using the variable assignments thus found, and the assignments for remaining variables, if any, are obtained by applying the DPLL procedure on the simplified formula.

(b) A satisfying assignment for ϕ_2 is obtained by applying the DPLL procedure. Formula ϕ_1 is then simplified using the variable assignments thus found. The assignments for remaining variables, if any, are obtained by applying the method for satisfiability checking of Horn formulae discussed in class on the simplified formula.

For each of the above approaches described above, determine whether it is guaranteed to give a satisfying assignment for ϕ for arbitrary Horn formula ϕ_1 and non-Horn CNF formula ϕ_2. If you think a particular approach will always lead to the correct answer, you must give justification (reasons) for the same. Else, you must give a counterexample to show that the approach may not lead to the correct answer. In each such case, you must also indicate which step in the Horn-formula satisfiabilty checking procedure and/or DPLL procedure causes the combined approach to fail in finding a satisfying assignment for ϕ.