CS206 Mid-Semster Examination

Max marks: 70 **Time: 3 hours**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others.***

1. *[10 marks]* Let $\{l_1, \ldots l_{10}\}$ be propositional literals (**not** variables), and let ϕ be the propositional logic formula $(l_1 \vee l_2) \wedge (l_3 \vee l_4) \wedge (l_5 \vee l_6) \wedge (l_7 \vee l_8) \wedge (l_9 \vee l_{10})$. Give a formula ψ (possibly involving additional literals beyond $\{l_1, \ldots l_{10}\}$), such that the following hold.

   (a) ψ is in disjunctive normal form (DNF)

   (b) ψ is valid if and only if ϕ is valid

   (c) ψ is a disjunction (sum) of no more than 25 conjuncts (products) of literals

   Note that there could be a pair of literals in $\{l_1, \ldots l_{10}\}$ such that one is the negation of the other. Consequently, it is not possible to determine the satisfiablity of ϕ without knowing what the literals are. Therefore, your solution **must not** involve determining the satisfiability of ϕ.

   To score full marks, you must briefly explain why your ψ is valid iff ϕ is valid.

2. *[7.5 + 7.5 marks]* Let $\mathsf{Cat}(x)$, $\mathsf{Dog}(x)$, $\mathsf{Striped}(x)$, $\mathsf{Friends}(x, y)$, $\mathsf{Equal}(x, y)$ and $\mathsf{ShortTempered}(x)$ be predicates to be evaluated over the set of all animals. The predicates have the following obvious interpretations. $\mathsf{Cat}(x)$ ($\mathsf{Dog}(x)$) evaluates to True iff x is a cat (dog). $\mathsf{Striped}(x)$ evaluates to True iff the coat of x is striped. $\mathsf{Friends}(x, y)$ evaluates to True iff x and y are friends; obviously, $\mathsf{Friends}(x, y)$ also implies $\mathsf{Friends}(y, x)$. $\mathsf{Equal}(x, y)$ evaluates to True iff x and y are one and the same animal. $\mathsf{ShortTempered}(x)$ evaluates to True iff x is short-tempered.

   Express the following English language sentences as predicate logic sentences (formulae without free variables). You may assume that the domain for evaluating the truth of these sentences is always the set of all animals (which could include animals other than cats and dogs as well).

   (a) There is a short-tempered dog who is not friendly with any other dog, but is friendly with at least one striped cat.

   (b) Every cat that is not striped is friendly with at least one dog that is not a friend of any striped cat.

3. *[5 + 10 + 10 marks]* In this question, we will reason about a proof system for propositional logic assuming that the only propositional connective is $\rightarrow$, and that the only propositional constant is $\perp$. For example, if x, y, z are propositional variables, then $(x \rightarrow (y \rightarrow \perp)) \rightarrow (\perp \rightarrow z)$ is a propositional logic formula using only the allowed connective and constant.

   (a) Let ϕ_1 and ϕ_2 be propositional logic formulae using $\rightarrow$ as the only connective and $\perp$ as the only constant. Give *semantically equivalent* formulae for $\phi_1 \wedge \phi_2$ and $\neg\phi_1$, such that $\rightarrow$ is the only connective and $\perp$ is the only constant in the resulting formulae. You must give justification for your claim of semantic equivalence.

(b) Your solution to the previous subquestion should convince you that any propositional logic formula can be converted to a semantically equivalent one using only $\to$ and $\bot$. A student now claims that it is possible to prove sequents in this version of propositional logic (with $\to$ as the only connective and $\bot$ as the only constant) using rules $\to_i$, $\to_e$, $\bot_e$ of the natural deduction system studied in class, in addition to the following special rule, called $(\to\bot)_e$ rule:

$$(\to \bot)_e : \ \frac{(\phi \to \bot) \to \psi,\ \phi \to \zeta,\ \psi \to \zeta}{\zeta}$$

Using only the above four rules, prove the following sequent:
$(\phi \to \bot) \to \psi,\ \phi \to \zeta \vdash (\psi \to \bot) \to \zeta$

(c) Do you think a proof system with only the above four rules, i.e. $\to_i$, $\to_e$, $\bot_e$ and $(\to \bot)_e$, is complete for the version of propositional logic that uses $\to$ as the only binary connective and $\bot$ as the only constant? In other words, given two formulas ϕ and ψ, each involving only $\to$ and $\bot$, such that $\phi \models \psi$, is it always possible to prove the sequent $\phi \vdash \psi$ using only the above four rules? Answers without justification will fetch zero marks.

4. *[10+10 marks]* Consider the following predicate logic sentences (formulae with no free variables):

$$\phi = \forall x(\exists y(E(x,y) \land \neg E(y,x)))$$
$$\psi = \forall x(\forall y(\forall z((E(x,y) \land E(y,z)) \to E(x,z)))).$$

We wish to evaluate these sentences (i.e., determine their truth value) over models obtained from directed graphs. Given a directed graph, $G = (V_G, E_G)$, a model $\mathcal{M}_G$ is obtained by letting the domain of variables be V_G, and by letting predicate $E(x,y)$ evaluate to true if and only if there is a directed edge from x to y in E_G. Since there are no function symbols and only one binary predicate symbol E in ϕ and ψ, every directed graph uniquely defines a model over which ϕ and ψ can be evaluated. For example, if G is the graph in Fig. 1, the corresponding model $\mathcal{M}_G$ has $\{v1, v2\}$ as the domain of variables, and $E(x,y) = \mathsf{True}$ if and only if $x = v1, y = v2$ or $x = y = v2$.

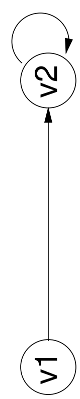

Figure 1:

(a) Give two directed graphs G_1 and G_2 with no more than 5 vertices in each such that ϕ evaluates to True over $\mathcal{M}_{G_1}$, and $\phi \to \psi$ evaluates to False over $\mathcal{M}_{G_2}$.

(b) We now wish to find a directed graph G such that $\phi \land \psi$ evaluates to True over $\mathcal{M}_G$. Indicate with justification whether G can be a finite graph (i.e., graph with finite number of vertices) and yet cause $\phi \land \psi$ to evaluate to True over $\mathcal{M}_G$. Simply answering "Yes" or "No" without justification will fetch zero marks.