CS226 Practice Problem Set 7 (Spring 2016) Solutions

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1. Let $X_1, \ldots X_6$ represent the next state values of the flip-flops whose current state values are $x_1, \ldots x_6$, respectively.

   Then, the entire state transition behaviour of the sequential circuit is described by the following formula, obtained from the given circuit diagram:

$$N(X_1, \ldots X_6, x_1, \ldots x_6, i) \quad = \quad (X_1 \leftrightarrow (x_6 \cdot i)) \cdot (X_2 \leftrightarrow x_1) \cdot (X_3 \leftrightarrow x_2) \cdot (X_4 \leftrightarrow x_3) \cdot (X_5 \leftrightarrow x_4) \cdot (X_6 \leftrightarrow x_5).$$

   Note that this formula has variables representing the primary inputs of the circuit, and also every flip-flop's next state and current state.

   The formula describing the set of initial states is $S_0(x_1, \ldots x_6) = (x_1 + x_2 + x_3) \cdot (x_4 + x_5 + x_6)$.

   For forward reachability analysis, recall from class that:

$$S_{i+1}(X_1, \ldots X_6) \quad = \quad S_0(X_1, \ldots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i\, (S_i(x_1, \ldots x_6) \cdot N(X_1, \ldots X_6, x_1, \ldots x_6, i))$$

   Note carefully the arguments of S_{i+1}, S_0 and S_i in the above equation. Refer to the slides used in class to understand this, if you have any doubts.

   To find the set of reachable states, we must iterate using the above equation until $S_i = S_{i+1}$ (as Boolean functions).

   Of course, S_{i+1} can be calculated using ROBDDs or AIGs, but we can also use the following simple observations, where $\varphi, \varphi_1, \varphi_2$ refer to arbitrary Boolean formulas, and u, v, w are Boolean variables (you should be knowing these from your course in logic):

   - $\varphi(u, v) \cdot (w \leftrightarrow u) \;\equiv\; \varphi(w, v) \cdot (w \leftrightarrow u)$.
   - $\exists u\, (\varphi_1 \cdot \varphi_2) \;\equiv\; \varphi_1 \cdot (\exists u\, \varphi_2)$, if the formula φ_1 doesn't refer to variable u.
   - $\exists u\, \varphi \;\equiv\; 1$, if the formula φ doesn't refer to the variable u.
   - $\exists u\, (u \leftrightarrow \varphi) \;\equiv\; 1$, if the formula φ doesn't refer to the variable u.
   - $\exists u\, (\varphi_1 + \varphi_2) \;\equiv\; (\exists u\, \varphi_1) + (\exists u\, \varphi_2)$
   - $\exists u\, \varphi(u, v) \;\equiv\; \varphi(0, v) + \varphi(1, v)$

   Using these observations, we get

$$\begin{aligned} S_{i+1}(X_1, \ldots X_6) &= S_0(X_1, \ldots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i\, (S_i(X_2, X_3, X_4, X_5, X_6, x_6) \cdot N(X_1, \ldots X_6, x_1, \ldots x_6, i)) \\ &= S_0(X_1, \ldots X_6) + \exists x_6 \exists i\, (S_i(X_2, X_3, X_4, X_5, X_6, x_6) \cdot (X_1 \leftrightarrow x_6 \cdot i)) \\ &= S_0(X_1, \ldots X_6) + S_i(X_2, X_3, X_4, X_5, X_6, 0) \cdot X_1' + S_i(X_2, X_3, X_4, X_5, X_6, 1) \cdot (X_1' + X_1) \\ &= S_0(X_1, \ldots X_6) + S_i(X_2, X_3, X_4, X_5, X_6, 0) \cdot X_1' + S_i(X_2, X_3, X_4, X_5, X_6, 1) \end{aligned}$$

   .

   Using the above formulation, we get

$$\begin{aligned} S_1(X_1, \ldots X_6) &= S_0(X_1, \ldots X_6) + (X_2 + X_3 + X_4) \cdot ((X_5 + X_6) \cdot X_1' + 1) \\ &= S_0(X_1, \ldots X_6) + (X_2 + X_3 + X_4) \cdot 1 \\ &= S_0(X_1, \ldots X_6) + (X_2 + X_3 + X_4) \end{aligned}$$

   With change of variables, we therefore have: $S_1(x_1, x_2, x_3, x_4, x_5, x_6) = ((x_1 + x_2 + x_3) \cdot (x_4 + x_5 + x_6)) + (x_2 + x_3 + x_4)$.

We can now obtain in a similar way the following:

$$
\begin{aligned}
S_2(X_1,\dots X_6) &= S_0(X_1,\dots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i \,(S_1(X_2,X_3,X_4,X_5,X_6,x_6)\cdot N(X_1,\dots X_6,x_1,\dots x_6,i))\\
&= S_0(X_1,\dots X_6) + (X_2+X_3+X_4) + (X_3+X_4+X_5)\\
&= S_0(X_1,\dots X_6) + (X_2+X_3+X_4+X_5)\\
S_3(X_1,\dots X_6) &= S_0(X_1,\dots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i \,(S_2(X_2,X_3,X_4,X_5,X_6,x_6)\cdot N(X_1,\dots X_6,x_1,\dots x_6,i))\\
&= S_0(X_1,\dots X_6) + (X_2+X_3+X_4) + (X_3+X_4+X_5+X_6)\\
&= S_0(X_1,\dots X_6) + (X_2+X_3+X_4+X_5+X_6)\\
&= ((X_1+X_2+X_3)\cdot(X_4+X_5+X_6)) + (X_2+X_3+X_4+X_5+X_6)\\
&= X_1+X_2+X_3+X_4+X_5+X_6\\
S_4(X_1,\dots X_6) &= S_0(X_1,\dots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i \,(S_3(X_2,X_3,X_4,X_5,X_6,x_6)\cdot N(X_1,\dots X_6,x_1,\dots x_6,i))\\
&= S_0(X_1,\dots X_6) + 1\\
&= 1\\
S_5(X_1,\dots X_6) &= S_0(X_1,\dots X_6) + \exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5 \exists x_6 \exists i \,(S_4(X_2,X_3,X_4,X_5,X_6,x_6)\cdot N(X_1,\dots X_6,x_1,\dots x_6,i))\\
&= 1
\end{aligned}
$$

Therefore, $S_5 = S_4$, and within 4 steps, all reachable states have been found. The set of reachable states is described by the Boolean formula $S_5 = 1$. Hence the set of unreachable states is represented by 0. In other words, there are no unreachable states.

2. Let $Z_0(x_1,\dots x_6)$ denote the set of "undesirable" states. Therefore, $Z_0(x_1,\dots x_6) = (x_1+x_2+x_3)\cdot(x_4+x_5+x_6)$.

   For backward reachability analysis, we need to use:

$$
Z_{i+1}(x_1,\dots z_6) = Z_0(x_1,\dots x_6) + (\exists X_1 \exists X_2 \exists X_3 \exists X_4 \exists X_5 \exists X_6 \exists i \; Z_i(X_1,\dots X_6)\cdot N(X_1,\dots X_6,x_1,\dots x_6,i))
$$

   Note carefully, once again, the arguments of Z_{i+1}, Z_0 and Z_i in the above equation. Refer to the slides used in class to understand this, if you have any doubts.

   Using the same observations as used in the previous sub-question, we get:

$$
\begin{aligned}
Z_{i+1}(x_1,\dots x_6) &= Z_0(x_1,\dots x_6) + \exists i \; Z_i(x_6\cdot i, x_1,x_2,x_3,x_4,x_5)\\
&= Z_0(x_1,\dots x_6) + Z_i(0,x_1,x_2,x_3,x_4,x_5) + Z_i(x_6,x_1,x_2,x_3,x_4,x_5)
\end{aligned}
$$

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   We can now use the above formulation to obtain the following:

$$
\begin{aligned}
Z_1(x_1,\dots x_6) &= Z_0(x_1,\dots x_6) + (x_1+x_2)\cdot(x_3+x_4+x_5) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5)\\
&= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5)\\
Z_2(x_1,\dots x_6) &= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_1)\cdot(x_2+x_3+x_4) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4)\\
&= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4)\\
Z_3(x_1,\dots x_6) &= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4) +\\
&\quad (x_4+x_5)\cdot(x_1+x_2+x_3) + (x_4+x_5+x_6)\cdot(x_1+x_2+x_3)\\
&= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4) +\\
&\quad (x_4+x_5+x_6)\cdot(x_1+x_2+x_3)\\
&= Z_0(x_1,\dots x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4)\\
&\quad \text{since } Z_0(x_1,\dots x_6) = (x_1+x_2+x_3)\cdot(x_4+x_5+x_6)\\
&= Z_2(x_1,\dots x_6)
\end{aligned}
$$

   Therefore, $Z_3 = Z_2 = (x_1+x_2+x_3)\cdot(x_4+x_5+x_6) + (x_6+x_1+x_2)\cdot(x_3+x_4+x_5) + (x_5+x_6+x_1)\cdot(x_2+x_3+x_4)$ gives the set of backward reachable states from Z_0. This describes the set of states from which an "undesirable" state described by $(x_1+x_2+x_3)\cdot(x_4+x_5+x_6)$ can be reached.

   Clearly, $x_1x_2x_3x_4x_5x_6 = 000000$ is not a satisfying assignment of $Z_3(x_1,\dots x_6)$. Therefore, no "undesirable" state can be reached, going forward from $x_1x_2x_3x_4x_5x_6 = 000000$. Of course, this would have been easy to see if we just tried to simulate the circuit starting from the state $x_1x_2x_3x_4x_5x_6 = 000000$. However, the question asked you to find the answer by finding the set of backward reachable states, and so that's how you should answer the question. The set of backward reachable states, once obtained, allows you to answer whether any "undesirable" state can be reached starting from **any** given initial state, not just from some easy initial state like $x_1x_2x_3x_4x_5x_6 = 000000$.