CS615 Autumn 2005 Quiz 1 Solution Sketch

1. The best way to go about problems of this type with upto three variables undergoing affine transformations, is to plot these variables in the $x-y$ or $x-y-z$ spaces. The transformations effected on the variables during the computation must then be visualized geometrically as transformations on shapes in these spaces. This often leads to a better intuition about invariants, post-conditions etc. when analyzing programs of the given type.

   Let ϕ_1 denote the predicate $(p \geq 0) \wedge (q \geq 0)$, and let ϕ_2 denote $(p \leq -1) \wedge (q \leq 0)$

   Let L^i denote the set of states when the program reaches the head of the while loop at the end of the i^{th} iteration. Using the axioms and inference rules of Hoare logic, we obtain the following:

   - $L^0 = \phi_1 \wedge (y = x) \wedge (x \geq 0) \wedge (y \geq 0)$
   - $L^1 = \phi_2 \wedge (1 \leq x \leq 101) \wedge (y = x - 1)$
   - $L^2 = \phi_1 \wedge (((q = 0) \wedge (0 \leq y \leq 99) \wedge (y = x + 1)) \vee ((q > 0) \wedge (1 \leq x \leq 100) \wedge (y = x - 1)))$
   - $L^3 = \phi_2 \wedge (((q = 0) \wedge (0 \leq y \leq 99) \wedge (y = x)) \vee ((q < 0) \wedge (2 \leq x \leq 101) \wedge (y = x - 2)))$
   - $L^4 = \phi_1 \wedge (((q = 0) \wedge (0 \leq y \leq 99) \wedge (y = x + 1)) \vee ((q > 0) \wedge (2 \leq x \leq 100) \wedge (y = x - 1)))$

   After this we converge, i.e. a previous L^i is obtained as L^5. In the above computation, the manipulation of linear inequalities and application of Hoare logic is fairly straightforward and has been skipped.

   Thus, the exact loop invariant is obtained as $L^0 \vee L^1 \vee L^2 \vee L^3 \vee L^4$. Since we have to express this invariant in the format required by the question, we can overapproximate the loop invariant as follows:

   $(\phi_1 \vee \phi_2) \wedge (((y = x) \wedge (x \geq 0) \wedge (y \geq 0)) \vee ((1 \leq x \leq 101) \wedge (y = x - 1)) \vee ((2 \leq x \leq 101) \wedge (y = x - 2)) \vee ((0 \leq y \leq 99) \wedge (y = x + 1)))$.

   However, this overapproximation is no longer a loop invariant. So we must propagate this overapproximation through the loop once more. This gives us one more line segment, $y = x - 3$, as a part of the loop invariant.

   We now take the convex hull of the line segments $y = x - 2$ and $y = x - 3$ to get $x - 3 \leq y \leq x - 2$. Then we also perform widening to ensure termination. After doing this, the loop invariant in the desired format is as follows: $(\phi_1 \vee \phi_2) \wedge ((y = x) \vee (y = x - 1) \vee (y = x + 1) \vee (x - 3 \leq y \leq x - 2))$