Max marks: 45 **CS615 Quiz 1** **Time: 1.5 hours**

- ***The exam is open book, notes and material brought to the exam hall.***
- ***Please state your assumptions clearly.***
- ***Results discussed in class may simply be cited, if needed. There is no need to prove such results in your answer book.***
- ***Do not copy from others or indulge in unfair means. Anybody found indulging in unfair means will be severely penalized, including being given the FR grade.***

In this question, we will reason about a program having two integer variables, x and y. We will use a new abstract domain, in which each element of the abstract lattice is a $4-$ tuple (a, b, c, d).
The abstraction and concretization function are defined as follows. In the following definitions, $\mathbf{Z}$ represents the set of integers and $\mathbf{Z}^+$ denotes $\mathbf{Z} \cup \{-\infty, +\infty\}$. In addition, we will assume that $-\infty < \infty$, $-\infty < n$ and $n < +\infty$ for all integers $n \in \mathbf{Z}$.

- $\alpha : \mathcal{P}(\mathbf{Z}^2) \rightarrow (\mathbf{Z}^+)^4$
  For all $S \subseteq \mathbf{Z}^2$, if $S \neq \emptyset$, $\alpha(S) = (a_S, b_S, c_S, d_S)$, where
  - $a_S = \min_{(x,y)\in S}(x - y)$
  - $b_S = \max_{(x,y)\in S}(x - y)$
  - $c_S = \min_{(x,y)\in S}(x + y)$
  - $d_S = \max_{(x,y)\in S}(x + y)$

  If $S = \emptyset$, $\alpha(S) = (+\infty, -\infty, +\infty, -\infty)$
- $\gamma : (\mathbf{Z}^+)^4 \rightarrow \mathcal{P}(\mathbf{Z}^2)$
  For all $(a, b, c, d) \in (\mathbf{Z}^+)^4$, if $a \leq b$ and $c \leq d$, $\gamma(a, b, c, d) = \{(x, y) \in \mathbf{Z}^2 \mid (a \leq x - y \leq b) \wedge (c \leq x + y \leq d)\}$.
  If either $a > b$ or $c > d$, $\gamma(a, b, c, d) = \emptyset$.

1. *[5 marks]* Let (a, b, c, d) be an abstract domain element such that $a \leq b$ and $c \leq d$. Find the strongest abstract postcondition of (a, b, c, d) with respect to the statement `x := 2 * x;` . In other words, if we were to interpret the single statement program `x := 2*x;` in the abstract domain described above, what will be the strongest postcondition if we start the program in the abstract state (a, b, c, d)?
   Your answer should be a four-tuple with each element of the tuple being an expression in a, b, c, d.

2. *[5 + 5 marks]* Similar to the above question, find the strongest abstract postcondition of (a, b, c, d) (assuming $a \leq b$ and $c \leq d$) with respect to each of the following statements:

   - `x := 2 + x;`
   - `y := y + x;`

3. *[5 + 5 + 5 marks]* Given two abstract domain elements (a_1, b_1, c_1, d_1) and (a_2, b_2, c_2, d_2), define suitable *lub*, *glb* and widen operators for the above abstract domain such that the analysis of the program in the next subquestion becomes as precise as you can make it. In each case, you must indicate an expression/give a short algorithm for computing each of the four components of the resulting abstract domain element.

4. *[15 marks]* Now consider the following program in our simple programming language:

```
L1: while (x+y <= 100) {
L2:     if (x+y <= 10)
L3:         x := 2*x;
L4:     else
L5:         x := 2 + x;
L6:     y := y + x;
```

   We wish to compute a loop invariant of the above program, assuming that the program is started in the abstract state $(0, 1, 0, 20)$. Using the expressions for abstract postconditions computed above, compute the above loop-invariant, using the following strategy:

   - In each iteration through the loop, compute the abstract domain element at `L6` as the *lub* of the abstract domain elements obtained as postconditions from the two branches of the `if else` statement.
   - Compute the loop invariant at `L1` by taking the *lub* at the loop head after the first iteration of the loop, and then using the widen operator for all subsequent iterations.