CS615 Quiz 1

Max marks: 30 **Time: 1.5 hours**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others.***

1. *[7.5 + 7.5 marks]* Let $\varphi = \exists k.\, ((k > 0) \wedge (x = 4^k) \wedge (y = 3.k))$ and $\psi = (\exists m.\, (x = 4.m) \wedge (y = 3^m))$.

   For each program P_i shown below, either (i) prove the validity of the Hoare triple $\{\varphi\}\ P_i\ \{\psi\}$, or (ii) demonstrate by a counterexample that $\{\varphi\}\ P_i\ \{\psi\}$ is not valid, i.e., give an initial state (values of x and y) that satisfies φ, but which is such that if the program P_i is started in this state, it terminates in a state that doesn't satisfy ψ.

   All variables in P_1 and P_2 are assumed to be of the `integer` type. All operations are assumed to return integers. The "x/y" operator returns the integer quotient obtained by dividing x by y. `func(x, y)` is assumed to be an unspecified function that takes two integer arguments and returns a Boolean result.

   If you are proving a Hoare triple, you must indicate invariants at each location of the program, all Hoare inference rules and any other axioms of arithmetic you might have used to arrive at your proof. If you are providing a counterexample, you must indicate an initial state, and the final state after termination of P_i and show that this state does not satisfy ψ. Your counterexample must work with any `func(x,y)` that returns a Boolean value. You may assume that during the execution of the while loops, `func(x,y)` will eventually return `false`, causing the loops to terminate.

```
P1:
   1:  while (x >  1) do
   2:      x := x/4;

   3:  while (func(x,y)) do
   4:      x := x + 4;
   5:      y := y*3;
```

```
P2:
   1: while (y > 3) do
   2:     y := y - 3;

   3: while (func(x, y)) do
   4:     x := x + 4;
   5:     y := y*3;
```

2. In this question, we wish to consider a new abstract domain – that of circles – for representing relations between a pair of integer valued variables. Each element in the abstract lattice is therefore a pair $\langle (a, b), r\rangle$, where (a, b) gives the co-ordinates of the centre of a circle in the $x - y$ plane, and r gives the radius of the circle.

   We wish to define the abstraction and concretization functions in the usual manner:

   - For a set S of points in the $x - y$ plane, $\alpha(S) = \langle (a, b), r\rangle$, such that the circle centred at (a, b) and with radius r is the smallest circle containing all points in S.
   - Given an abstract element $A = \langle (a, b), r\rangle$ in the abstract lattice, $\gamma(A) = \{(x, y) \mid (x - a)^2 + (y - b)^2 \leq r^2\}$.

We also wish to define an ordering relation in the abstract domain in the usual way, i.e. $A = \langle (a_1, b_1), r_1 \rangle \sqsubseteq \langle (a_2, b_2), r_2 \rangle = B$, we say $A \sqsubseteq B$ iff $\gamma(A) \subseteq \gamma(B)$.

(a) *[5 marks]* Does the ordering relation defined above qualify to be a partial order? If not, indicate how you would canonicalize elements in the abstract domain, so that we get a partial order from the above ordering relation. Otherwise, show that the above ordering relation is reflexive, anti-symmetric and transitive.

(b) *[5 marks]* Indicate how you would construct an *lub* operator in the abstract domain of circles. In other words, given $A = \langle (a_1, b_1), r_1 \rangle$ and $B = \langle (a_2, b_2), r_2 \rangle$, if $C = \langle (a_3, b_3), r_3 \rangle = A \sqcup B$, express each of a_3, b_3 and r_3 in terms of $a_1, b_1, r_1, a_2, b_2, r_2$.

(c) *[5 marks]* Indicate how you would construct a *glb* operator in the abstract domain of circles. In other words, given $A = \langle (a_1, b_1), r_1 \rangle$ and $B = \langle (a_2, b_2), r_2 \rangle$, if $C = \langle (a_3, b_3), r_3 \rangle = A \sqcap B$, express each of a_3, b_3 and r_3 in terms of $a_1, b_1, r_1, a_2, b_2, r_2$.