CS615 Theory Quiz 2 (Autumn 2017)

Max marks: 25 **Time: 60 mins**

- *Be brief, complete and stick to what has been asked.*
- *Unless asked for explicitly, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others. Penalty for offenders: FR grade.***

1. For purposes of this question, assume all program variables are of type `int`. Further assume that $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}$ denotes the set of integers.

   We wish to define a *non-relational* abstract domain where the set of concrete values of an individual variable is abstracted by a pair of integers $\langle a, b\rangle$, with $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. Furthermore, we require that if $a \neq 0$, then $b < a$.

   The abstraction of a program state is simply the tuple of abstractions of individual program variables. Therefore, we will concern ourselves with simply the abstraction of a single program variable in this question (like in the case of intervals).

   The abstraction and concretization functions for a single program variable are defined as follows:

   - $\gamma(\langle a, b\rangle) = \{n \mid \exists m \in \mathbb{Z},\, n = a.m + b\}$ for every $a \in \mathbb{N}$ and $b \in \mathbb{Z}$.
   - $\alpha(S) = \langle a, b\rangle$, such that
     - $a = \gcd\{|n - n'| \; s.t. \; n, n' \in S\}$. Assume $\gcd(0, n) = 1$ for all $n \in \mathbb{Z}$.
     - $b = \min\{n \mid n \in S\}$

   The top-element of the abstract lattice (for a single program variable) is assumed to be $\langle 1, 0\rangle$. Note that $\gamma(\langle 1, 0\rangle) = \mathbb{Z}$. The bottom element is assumed to be a special element $\perp$ such that $\alpha(\emptyset) = \perp$ and $\gamma(\perp) = \emptyset$.

   (a) *[5 marks]* It is easy to show that α and γ forms a Galois connection. Does it also form a *Galois insertion*, i.e. is it the case that $\forall \langle a, b\rangle \in \mathbb{N} \times \mathbb{Z},\ \alpha(\gamma(\langle a, b\rangle)) = \langle a, b\rangle$? You may assume that if $a \neq 0$, then $b < a$. Either give a proof of the pair (α, γ) being a Galois insertion, or provide a counterexample.

   (b) *[5+5+5+5 marks]* Give as good *effective* definitions as you can for the following abstract domain operators. In each case, your definition should be *effective*, i.e. it should be possible to write an algorithm (that necessarily terminates, of course) that computes the resulting given the input abstract values.

      i. Partial order relation ($\sqsubseteq$)
      ii. Least upper bound ($\sqcup$)
      iii. Greatest lower bound ($\sqcap$)
      iv. Widen (∇)

   Give brief explanation in each case why you think your definition is as good as possible.