CS719 Homework 1

Due date: Aug 19, 2011, 5pm

1. Use natural deduction to prove the sequents enumerated as (a) and (b) below. You must indicate which proof rule you are applying at each step, as shown in the example proof below.

   Example: We want to prove the sequent:

   $$\forall x. (P(x) \to Q(x)) \;\; \exists x. P(x) \;\vdash\; \exists x. Q(x).$$

   Proof:

1.	$\forall x. (P(x) \to Q(x))$	......	Premise
2.	$\exists x. P(x)$	......	Premise
3.	$x_0 \quad P(x_0)$	......	Let x_0 be a value and assume $P(x_0)$
4.	$P(x_0) \to Q(x_0)$	......	$\forall_e$ on 1
5.	$Q(x_0)$	......	$\to_e$ on 3, 4
6.	$\exists x. Q(x)$	......	$\exists_i$ on 5
7.	$\exists x. Q(x)$	......	$\exists_e$ on 2 and 3 – 6

   (a) *[5 marks]* $\forall x \forall y. ((P(x) \vee Q(y) \wedge (Q(x) \vee \neg P(y)) \;\vdash\; \forall x. Q(x)$

   (b) *[5 marks]* $\vdash\; \exists y. ((\forall x. P(x)) \;\to\; P(y))$

2. We have seen in class that first-order sentences on a vocabulary consisting of a single binary predicate E (and possibly also using equality) can be used to describe several interesting properties of graphs. Show that there exists a first-order sentence ϕ on the vocabulary $\{E\}$ (and possibly also using equality) that characterizes each of the following classes of graphs. Thus, ϕ should be such that every graph belonging to the class under consideration is a model of ϕ, and every model of ϕ can be viewed as a graph belonging to the class under consideration.

   (a) *[5 marks]* Rooted trees with depth (i.e. no. of edges along any path from the root to a leaf) ≤ 3. Assume that each edge in a tree points from a node to its children.

   (b) *[5 marks]* Directed acyclic graphs with diameter (i.e. length of the longest directed path between any pair of vertices) ≤ 3.

   (c) *[5 marks]* Graphs with no vertex covers of size ≤ 3.

3. Taking cue from the formulas you have written as part of your solution for the previous problem, either show that the following sets of graphs are inexpressible in first-order logic over the vocabulary consisting of a binary predicate and possibly equality, or give a first-order logic formula characterizing the corresponding set of graphs.

   (a) *[5 marks]* The set of all finite rooted trees.

   (b) *[5 marks]* The set of all infinite rooted trees.

   (c) *[5 marks]* The set of all finite or infinite rooted trees.