Four Colour Theorem

A Computational View

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Four Colour Theorem

Theorem

Every planar map is four colourable.

- Planar map: A partition of the plane into a finite number of regions bounded by simple closed curves.
- Four colourable: Each region can be coloured using one of four colours so that any two regions that share a boundary of non-zero length have different colours.

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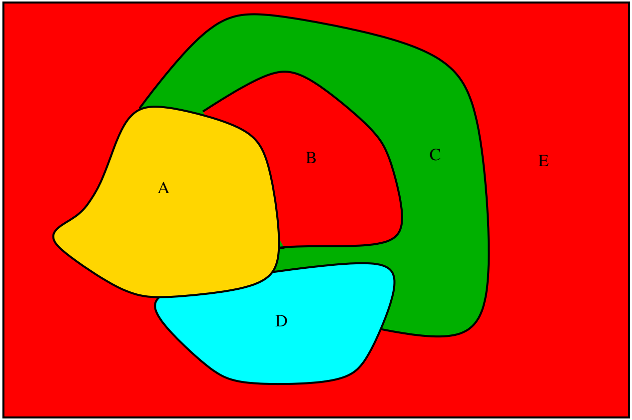
A Map



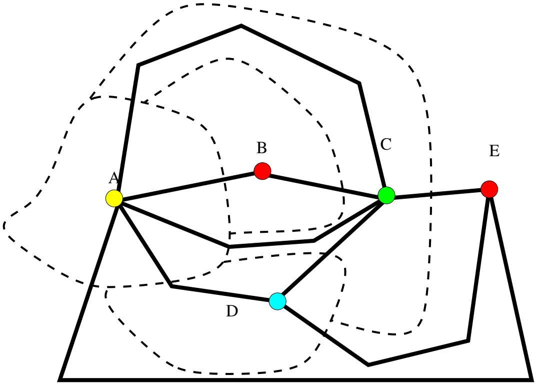
What is different about it?

- Easy to state and understand, first conjectured by an undergraduate student (1852).
- Several failed attempts, including erroneous published proofs.
- The known proofs all use a computer to verify some cases.
- Lead to major developments in graph theory, several conjectured and proven generalizations within graph theory.
- Reformulated in terms of many other mathematical objects, can be viewed in different ways.
- Still hope of finding a simple, easily verifiable proof.

Plane Graphs

- Represent each region by a point contained in the region.
- If two regions share a boundary draw a simple curve joining the points representing them.
- Points representing regions are vertices and curves joining them are edges.
- Edges are drawn so that their interiors are disjoint.
- This gives a plane graph.
- This is also a planar map called the dual of the original map.

Dual Map



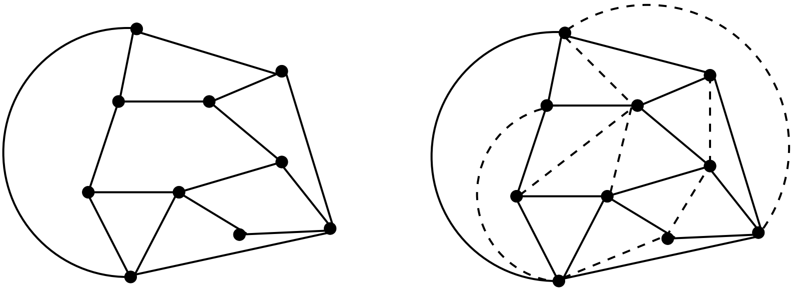
Four Colouring Plane Graphs

- Equivalent statement of the Four Colour Theorem.
- **Every plane graph is four colourable.**
- Each vertex is assigned one of four colours.
- Vertices joined by an edge must have different colours.
- There exists such an assignment of colours for every plane graph.

Plane Triangulations

- Given any plane graph, add as many edges to it as possible, as long as the graph remains a plane graph.
- If the graph obtained after adding edges in four colourable then so is the original graph.
- No more edges can be added to a plane graph keeping planarity if an only if every region, also called a face, is bounded by exactly three edges.
- Such a plane graph is called a plane triangulation.
- Sufficient to prove that plane triangulations are four colourable.

Triangulating a Plane Graph



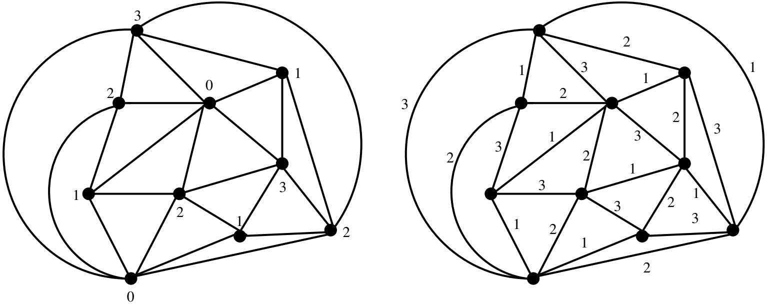
Tait's reformulation

- Suppose a plane triangulation is four colourable.
- Assume the four colours are **0, 1, 2, 3**.
- Assign to each edge the XOR of the colours assigned to its endpoints.
- Four colouring implies every edge gets one of the three colours **1, 2, 3**.
- The XOR of colours of edges in any cycle, and more generally, any subgraph with all vertices of even degree, must be **0**.
- All three edges on the boundary of any face get distinct colours.

Three Colouring Edges

- Converse holds for plane triangulations.
- Consider a colouring of edges with colours **1,2,3** such that the boundary of every face has distinct colours.
- The XOR of the colours on the boundary of any face is **0**.
- Any cycle, considered as a set of edges, is the XOR of the boundaries of faces contained inside it.
- For any cycle, and hence for any even subgraph, the XOR of the edge colours is **0**.
- For any two vertices, all paths between them must have the same XOR of edge colours.
- Assign colour **0** to a fixed vertex r, and colour any other vertex by the XOR of the edge colours in any path from r.
- This gives a four colouring.

Three Edge Colouring



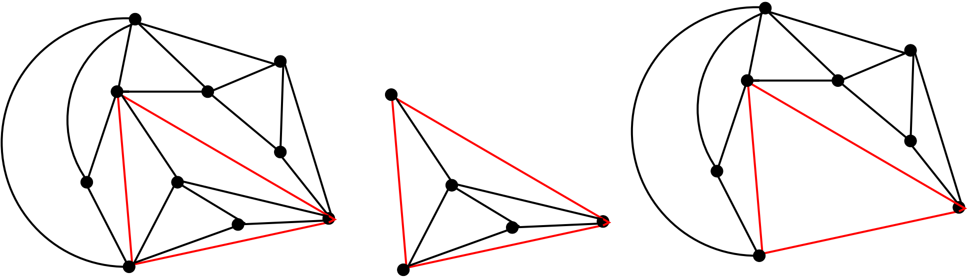
Heawood's reformulation

- Consider an edge colouring with three colours such that the boundary of any face has distinct colours.
- Assign value $+1$ to an internal face if the colours **1,2,3** appear on the face in clockwise order and -1 otherwise.
- For every internal vertex v the sum of values of faces whose boundary contains v is divisible by 3.
- The converse is also true, given such an assignment of $+1$ or -1 values to the faces, a three edge coloring with distinct colours on the boundary of each face can be constructed.

Separating Triangles

- A triangle in a plane triangulation is called a separating triangle if there are vertices of the graph in its interior and exterior.
- If there exists a separating triangle, induction can be used easily.
- Delete vertices in the exterior to get a smaller triangulation and the vertices in the interior to get another smaller triangulation.
- By induction, both are four colourable.
- In any four colouring of both triangulations, the vertices of the separating triangle get three distinct colours, and may be assumed to be same in both, by permuting the colours.
- The colourings can be combined to get a four colouring of the original triangulation.

Separating Triangle



Whitney's Theorem

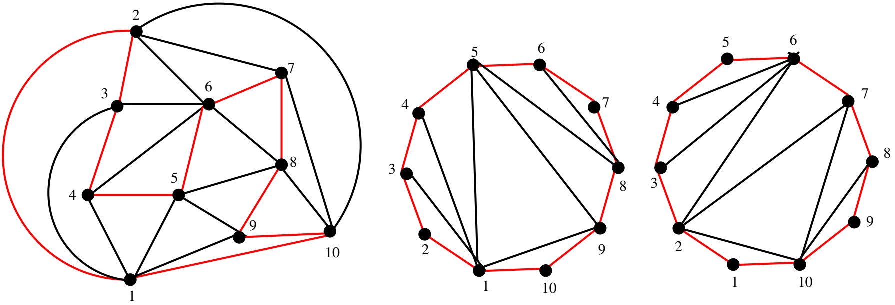
- Sufficient to consider triangulations without a separating triangle.

Theorem (Whitney)

Every plane triangulation without a separating triangle has a Hamiltonian cycle, that is, a cycle passing through all vertices.

- The edges of the triangulation are divided into those inside, on or outside the Hamiltonian cycle.
- Fixing the colours of edges on the Hamiltonian cycle, fixes the colours of edges inside and outside the cycle uniquely, if a three edge colouring is possible.
- The problem reduces to finding a colouring of the edges in the cycle, so that it can be extended to the edges inside and outside the cycle.

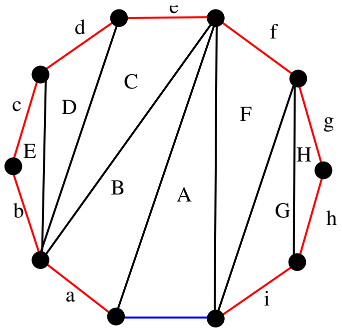
Hamiltonian cycle

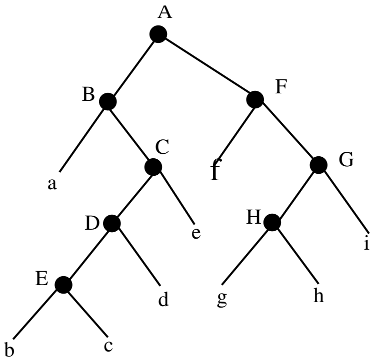


Binary Trees

- Triangulations of a cycle represented by binary trees.
- Fix an edge of the cycle as the base.
- Triangle containing the base is the root node of the binary tree.
- Divides the cycle into two edge-disjoint cycles.
- The left (right) subtree is the binary tree corresponding to the cycle with the left (right) edge of the root triangle as the base.
- Every triangle corresponds to an internal node.
- Every edge of the cycle other than the base corresponds to an external node.

Triangulation as a Binary Tree

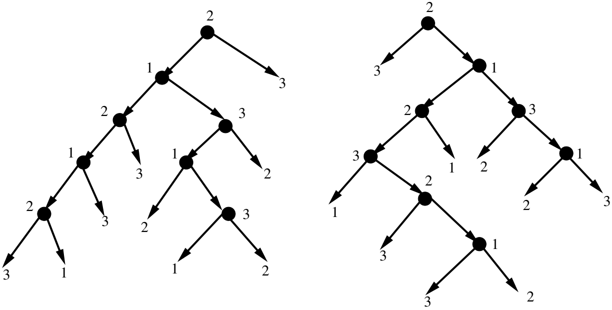




Colouring Binary Trees

- Two arbitrary binary trees corresponding to the triangulation inside and outside the Hamiltonian cycle.
- Colouring edges is equivalent to colouring nodes of the binary trees.
- If an internal node is coloured **i** then its left and right child must be coloured **j, k** where { **i,j,k** } = { **1,2,3** }.
- Corresponding external nodes and the root node must get the same colour in both trees.
- **Four colour theorem equivalent to the statement that this is always possible for any pair of binary trees.**

Colouring Binary Trees



3 1 3 3 2 1 2 2 3

Context Free Grammar

- Expressed using the context free grammar

$$\begin{array}{lll} \mathbf{1} & \rightarrow & \mathbf{23} \mid \mathbf{32} \mid 1 \\ \mathbf{2} & \rightarrow & \mathbf{13} \mid \mathbf{31} \mid 2 \\ \mathbf{3} & \rightarrow & \mathbf{12} \mid \mathbf{21} \mid 3 \end{array}$$

- Bold numbers are non-terminals, start symbol is **1**.
- A string in the language can have many parse trees.
- **Given any two binary trees, there exists a string in the language that can be parsed using both the trees.**
- Can be proved easily if the trees have some structure, for example linear trees.

A Problem on Strings

- Alphabet of four letters { *a,b,c,d* } or {*A, C, G, T* }, the building blocks of the genome sequence.
- A string is a sequence of these letters, the length is the number of letters.
- Given a string $s = s_1 s_2 s_3 \dots s_l$, the string $s_i s_{i+1} \dots s_j$ for $1 \leq i \leq j \leq l$ is a substring.

An Automaton

- A state is a set $S(l)$ of strings, all of the same length $l \geq 2$.
- A transition is defined by a pair of numbers i, j such that $1 \leq i < j \leq l$.
- The new state is the set of strings T of length $l + i + 2 - j$ obtained as follows:
  - For every string $s \in S(l)$, replace the substring $s_{i+1} \ldots s_{j-1}$ by any single letter that does not occur in the substring and is not equal to either s_i or s_j.
  - If there is more than one such letter, put all possible such strings in T.
  - If there is no such letter, the string s does not contribute any string to T.
- The set T may be empty.
- If not empty, every string in T has length $l + i + 2 - j$ and further transitions can occur from it.

Sample Transitions

- $\{acb\} \xrightarrow{[1,2]} \{abcb, adcb\}$
- $\{abcb, adcb\} \xrightarrow{[2,4]} \{abab, abdb, adab\}$
- $\{abab\} \xrightarrow{[1,3]} \{acab, adab\} \xrightarrow{[2,4]} \{acdb, adcb\} \xrightarrow{[1,4]} \emptyset$
- $\{abab, abdb\} \xrightarrow{[1,3]} \{acab, adab, acdb\} \xrightarrow{[2,4]}$
  $\{acdb, adcb, acab\} \xrightarrow{[1,4]} \{adb\}$

The Formulation

Theorem

The four colour theorem is true if and only if there is no sequence of transitions from the set {acb } to the empty set.

- Every sequence of transitions starting from {acb } corresponds to a plane triangulation and vice versa.
- The triangulation is not four colourable if and only if the sequence of transitions leads to the empty set.

The Formulation

Theorem

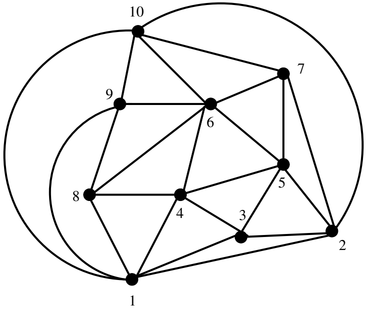
The four colour theorem is true if and only if there is no sequence of transitions from the set {acb } to the empty set.

- Every sequence of transitions starting from $\{acb\}$ corresponds to a plane triangulation and vice versa.
- The triangulation is not four colourable if and only if the sequence of transitions leads to the empty set.

Equivalence

- Every triangulation can be built up starting with a triangle and adding one vertex at a time.
- At every step, all internal faces are triangles, the external face may be of any length. Such graphs are called near-triangulations.
- Every new vertex added is adjacent to a consecutive sequence of vertices on the current external face.
- The current state represents the possible sequence of colours on the outer boundary in a proper four colouring of the current near-triangulation.
- Addition of a vertex corresponds to state transition.

Triangulation and Transitions



Boundary	Transitions
1 2	–
1 3 2	–
1 4 3 2	[1,2]
1 4 5 2	[2,4]
1 4 6 5 2	[2,3]
1 4 6 7 2	[3,5]
1 8 6 7 2	[1,3]
1 9 6 7 2	[1,3]
1 10 2	[1,5]

Generalization

- **Decision Problem:** Does there exist an algorithm to decide if the empty set can be reached by a sequence of transitions starting from a given set of strings of equal length?
- If so, this gives a short proof of the Four Colour Theorem, just apply the algorithm to the set $\{acb\}$.
- **Conjecture:** There exists a computable function $f(l)$, perhaps even linear, such that for any set of strings of length l, if the empty set is reachable, then it is reachable in at most $f(l)$ transitions.
- If true, this gives a simple decision algorithm, just consider all sequences of transitions of length at most $f(l)$.

Variations

- Many variations possible.
- Use different number of letters and different starting sets.
- **Problem:** Every planar graph can be coloured with $k + 1$ colours such that the vertices of any specified cycle of length k get distinct colours for all $k \geq 3$.
- Corresponds to using an alphabet with $k + 1$ letters and an initial string with k distinct letters.
- Use different rules for generating the new set.
- Allows considering colourings of plane triangulations with different properties.

Thank you

Questions?

What are the applications? - None!