Chords of 2-factors in planar cubic bridgeless graphs

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Abstract

We show that every edge in a 2-edge-connected planar cubic graph is either contained in a 2-edge-cut or is a chord of some cycle that is contained in a 2-factor of the graph. As a consequence, we show that every edge in a cyclically 4-edge-connected planar cubic graph, except K_2^3 and K_4, is contained in a perfect matching whose removal disconnects the graph. We obtain a complete characterization of 2-edge-connected planar cubic graphs that have an edge such that every 2-factor containing the edge is a Hamiltonian cycle, and also of those that have an edge such that the complement of every perfect matching containing the edge is a Hamiltonian cycle. Another immediate consequence of the main result is that for any two edges contained in a facial cycle of a 2-edge-connected planar cubic graph, there exists a 2-factor in the graph such that both edges are contained in the same cycle of the 2-factor. We conjecture that this property holds for any two edges in a 2-edge-connected planar cubic graph, and prove it for planar cubic bipartite graphs. The main result is proved in the dual form by showing that every plane triangulation admits a vertex 3-coloring such that no face is monochromatic and there is exactly one specified edge between a specified pair of color classes.

1 Introduction

A classical result in graph theory is Petersen's theorem [8] that every 2-edge-connected cubic graph has a perfect matching, and hence a 2-factor obtained by taking the complement of the matching. This can be strengthened to show that there exists a perfect matching including any specified edge, and a 2-factor including any two specified edges in the graph [9]. More can be said about the structure of the 2-factor if the cubic graph has additional properties, in particular, for 2-edge-connected planar cubic graphs. It is well-known that the four color theorem is equivalent to the statement that every 2-edge-connected planar cubic graph is 3-edge-colorable. This is equivalent to the statement that every such graph has a 2-factor in which all cycles have even lengths. It was shown in [2] that every 2-edge-connected planar cubic graph with at least 6 vertices has a *disconnected* 2-factor, that is, a 2-factor with more than one component. Häggkvist [5] showed that every edge in a *bipartite* cubic graph is either contained in a 2-edge-cut or is a chord of some cycle that is contained in a 2-factor of the graph. Barnette [1] conjectured that every 3-edge-connected planar cubic bipartite graph has a Hamiltonian cycle, which is a connected 2-factor, and this question is still open.

In this paper, we show that Häggkvist's result also holds for all 2-edge-connected planar cubic graphs. In other words, every edge in a 2-edge-connected planar cubic graph is either contained in a 2-edge-cut or is a chord of some cycle that is contained in a 2-factor in the graph. An edge in a cubic graph is a chord of some cycle iff it is not contained in an edge-cut of size at most two. Thus another way of saying this is that if an edge is a chord of some cycle then it is a chord of some cycle that is contained in a 2-factor. As a consequence, we get a simpler and completely different proof of the result in [2] and in fact prove much stronger statements. We show that every edge in a cyclically 4-edge-connected planar cubic graph, except K_2^3 and K_4, is contained in a perfect matching whose complement is a disconnected 2-factor. We call such a perfect matching a *separating* perfect matching. We obtain a complete characterization of 2-edge-connected planar cubic graphs that have an edge that is not contained in a separating perfect matching, and also of those 2-edge-connected planar cubic graphs that have an edge such that every 2-factor containing the edge is a Hamiltonian cycle. It may be noted that a characterization of cubic bipartite graphs in which every 2-factor is a Hamiltonian cycle has been conjectured [4], but the problem is still open.

Another consequence of our result is that for any two edges contained in a facial cycle of a 2-edge-connected planar cubic graph, there exists a cycle containing both the edges that is contained in a 2-factor of the graph. We conjecture that this property holds for any two edges in the graph. It is also possible that it holds for any two edges in a cubic bipartite graph. We prove it for any two edges in a planar cubic bipartite graph.

The Petersen graph, which is the smallest 2-edge-connected cubic graph that is not 3-edge-colorable, also shows that in general, a 2-edge-connected cubic graph may not have any 2-factor in which some cycle has a chord. In this case, no edge is contained in a 2-edge-cut and no edge is a chord of a cycle contained in a 2-factor. It is an interesting question to determine which 2-edge-connected cubic graphs have a 2-factor in which some cycle has a chord. It is possible that every 3-edge-colorable cubic graph has a 2-factor in which some cycle has a chord, although it is not true that every edge that is not contained in a 2-edge-cut is a chord of some cycle contained in a 2-factor in such a graph. It has been conjectured [6] that every 3-edge-colorable cubic graph has an edge e such that the graph obtained by deleting e and contracting one other edge incident with each endpoint of e is also 3-edge-colorable. Perhaps a stronger statement that combines both properties may hold. If every 3-edge-colorable cubic graph has a 3-edge-coloring such that some 2-edge-colored cycle has a chord, it would imply both the results. This statement can be verified easily for planar cubic graphs, even without assuming that all of them are 3-edge-colorable. It also holds for cubic bipartite graphs by Häggkvist's result, since any such graph must have an edge not contained in a 2-edge-cut.

One reason for studying 2-factors in 2-edge-connected planar cubic graphs is that results on these have a dual version in terms of vertex colorings of the dual graphs. The planar duals of 2-edge-connected planar cubic graphs are plane triangulations, which are loopless plane graphs embedded in the plane so that each face has 3 edges on its boundary. The most famous example of this is of course Tait's reformulation of the four color theorem. Tait in fact falsely conjectured that all 3-edge-connected planar cubic graphs are Hamiltonian, which would have proved the four color theorem. Penaud [7] first noted that the existence of a 2-factor in a 2-edge-connected planar cubic graph implies that any plane triangulation has a non-monochromatic 2-coloring, that is a coloring of the vertices with 2 colors such that no three vertices that are on the boundary of the same face have the same color. As shown in [3], the existence of a disconnected 2-factor in a 2-edge-connected planar cubic graph implies the existence of a strict 3-coloring of the vertices of a plane triangulation such that no face is monochromatic and also no face is rainbow, that is, some two vertices in the boundary of any face have the same color. In general, it is shown in [3] that the existence of a 2-factor with at least k components is equivalent to the existence of a strict $(k+1)$-coloring of the dual such that no face is monochromatic or rainbow. Although this problem is NP-complete for arbitrary k, it is still open for any fixed $k \geq 3$.

The dual version of finding a 2-factor with a chord is to find a non-monochromatic 2-coloring of the vertices of a triangulation such that the subgraph formed by the monochromatic edges has a bridge. This is equivalent to finding a non-monochromatic 3-coloring such that for some two color classes there is exactly one edge (the bridge) between the color classes. To find a 2-factor in which a specified edge is a chord of some cycle, we fix the edge between the color classes to be the dual of the edge required to be a chord. We will use this dual formulation and prove the existence of such a 3-coloring by induction. The dual result may also be of independent interest.

2 Terminology

The notation and terminology used is mostly standard. We will only clarify the terms that are specific to this work. We consider undirected graphs that may have multiple edges but no self-loops. A graph is k-edge-connected if it cannot be disconnected by removing less than k edges. An edge whose removal increases the number of connected components in a graph is called a bridge. A graph is cubic if the degree of every vertex is three. The cubic graph with two vertices and three edges joining them is denoted by K_2^3. A cubic graph is said to be cyclically k-edge-connected if removing any set of less than k edges results in a graph with at most one component containing a cycle. A plane graph is a graph that has been embedded in the plane. Any such embedding divides the plane into connected regions called faces. The unbounded region will be called the external face and all other faces are said to be internal. We will identify a face by the sequence of vertices and edges on its boundary in circular order. A plane triangulation is a plane graph such that every face is bounded by 3 edges. The dual of any plane cubic graph is a plane triangulation and vice-versa. Since we consider only bridgeless cubic graphs, their duals will be triangulations without any self-loops. This implies the plane triangulations are necessarily 2-connected and no vertex or edge is repeated on the boundary of any face. A plane near-triangulation is a 2-connected plane graph such that every internal face is bounded by 3 edges. The boundary of a near-triangulation is the sequence of vertices that are on the boundary of the external face, in circular order. The vertices and edges that are on the boundary of the external face are called boundary vertices and edges, respectively, and the other vertices and edges are said to be internal. A chord of a near-triangulation is an internal edge both of whose endpoints are boundary vertices.

A 3-coloring of a graph G is a function $f : V(G) \to \{a, b, c\}$ that assigns one of 3 colors to each vertex in G. An internal face of a near-triangulation is said to be monochromatic in a 3-coloring f if all 3 vertices on the boundary of the face have the same color. The coloring f is said to be non-monochromatic if there is no monochromatic internal face in f. The coloring is said to be a 2-coloring with colors $\{a, c\}$ if $f(v) \neq b$ for all $v \in V(G)$.

A non-monochromatic 3-coloring f of a near-triangulation G with boundary $v_1, v_2, \ldots, v_l$ is said to be *special* if for any two vertices u, v such that $f(u) = a$ and $f(v) = b$, u is adjacent to v iff $u = v_1$ and $v = v_2$. An adjacent pair of vertices u, v is called an ab-pair in a coloring f if $f(u) = a$ and $f(v) = b$. Thus a 3-coloring of G is special if it is non-monochromatic and the only ab-pair is v_1, v_2. For a string $s = s_1 s_2 \ldots s_l$ of length l over the alphabet $\{a, b, c\}$, we say that G has a special 3-coloring f with colors s assigned to the boundary if f is a special 3-coloring of G such that $f(v_i) = s_i$ for $1 \leq i \leq l$. We say that s is *feasible* for G if there exists a special 3-coloring of G with colors s assigned to the boundary. Note that we can choose the vertex v_1 in the boundary arbitrarily and label the other vertices in circular order, either clockwise or anti-clockwise. In the case of a triangulation, we can choose the unbounded face also arbitrarily, so that any pair of adjacent vertices in a triangulation may be labeled v_1, v_2.

We now state some basic results using the above terminology.

Lemma 1 (Penaud) *Let G be a plane triangulation without self-loops having the boundary v_1, v_2, v_3. Then there exists a non-monochromatic 2-coloring of G with colors $\{a, c\}$ such that the boundary is assigned colors aac.*

Proof: This follows from the stronger form of Petersen's theorem. Consider a matching in the dual that contains the dual edge of $v_1 v_2$. Deleting the edges in this matching gives a 2-factor which is a collection of disjoint cycles in the plane. The regions into which the plane is divided by these cycles can be 2 colored so that adjacent regions get distinct colors. Assign a color to a region depending on the parity of the number of cycles in the 2-factor whose interior contains the region. Assign a vertex in the triangulation the color of the region containing the face in the dual corresponding to the vertex. Then no face in the triangulation will be monochromatic since the vertex in the dual corresponding to the face will be contained in some cycle in the 2-factor, and all 3 faces incident with the vertex cannot be on the same side of the cycle. However, they will be on the same side of any other cycle, which implies all three cannot have the same color. Since the dual edge of $v_1 v_2$ is in the matching, v_1 and v_2 will get the same color. We can assume this color is a, without loss of generality, and since v_1, v_2, v_3 is a face in G, v_3 must have a different color. □

A small modification of this gives the coloring equivalent of a 2-factor with a chord.

Lemma 2 *Let G be a 2-edge-connected plane cubic graph and uv an edge in G that is not contained in a 2-edge-cut. Let G' be the dual of G with the endpoints of the dual edge of uv labeled as v_1 and v_2. Then G has a 2-factor in which some cycle has uv as a chord iff the dual G' has a special 3-coloring.*

Proof: Suppose G has such a 2-factor. We first assign colors a, c as in the proof of Lemma 1 so that no face in G' is monochromatic. Let C be the cycle in the 2-factor such that uv is a chord of C. Without loss of generality, we can assume that the chord uv is in the interior of C and the region R in the interior of C that has C as part of its boundary is colored a. In particular, this implies v_1, v_2 are colored a. The chord uv splits R into two regions, R_1 which contains the face corresponding to v_1 and R_2 which contains the face corresponding to v_2. We now change the color of all vertices in G' corresponding to faces contained in the region R_2 to b. This gives a special 3-coloring of the dual triangulation G'. Conversely, if there exists a special 3-coloring of G', the edges in G corresponding to the edges in G' whose endpoints have the same color, and the edge uv, form a matching in G. Since there is no 2-edge-cut in G containing the edge uv, there is a single edge between v_1 and v_2 in G'. This edge must be a bridge in the subgraph of G' containing the edges having endpoints of the same color and the edge v_1v_2, since deleting it separates v_1 from v_2. Therefore the dual edge uv must be a chord of some cycle, since it must become a self-loop after contracting all the edges in the 2-factor. □

We will be working with plane triangulations and we define some simple operations on them. Let uv be an edge in a plane triangulation G. Let w_1, w_2 be the vertices such that u, v, w_1 and u, v, w_2 are the two faces in G whose boundary contains the edge uv. Suppose $w_1 \neq w_2$. The operation of *flipping* the edge uv replaces the edge uv by the edge w_1w_2 on the same side of the cycle u, w_1, v, w_2 that contains the edge uv. Note that the result of this operation is again a triangulation. The operation of *contracting* the edge uv to the vertex u deletes the edges vw_1, vw_2 and uv and merges the vertex v with the vertex u. All the other edges incident with v are now moved to u, preserving the circular order of the edges at all vertices and thus the embedding. If there was another edge between u and v, it will now become a self-loop at u. The only difference between ordinary contraction of an edge and this is that we delete the edges vw_1 and vw_2, so that all faces in the resulting graph are still bounded by 3 edges. In the dual, this is equivalent to deleting an edge and suppressing the resulting degree 2 vertices by contracting an edge incident with them. Again, with this definition of contraction, the result of applying this operation is a triangulation. The operations are also defined in the same way for internal edges of near-triangulations.

3 Main Result

In this section we prove the main theorem.

Theorem 1 *Every edge in a 2-edge-connected planar cubic graph is either contained in a 2-edge-cut or is a chord of some cycle that is contained in a 2-factor of the graph.*

We will actually prove the following reformulation, whose equivalence follows from Lemma 2.

Theorem 2 *Let G be a plane triangulation with the boundary v_1, v_2, v_3. Then there exists a special 3-coloring of G with the boundary assigned the colors abc.*

In order to prove Theorem 2, we need to prove another statement, involving near-triangulations with a 4-sided boundary, simultaneously by induction.

Lemma 3 *Let G be a near-triangulation with boundary v_1, v_2, v_3, v_4. Then at least two of the assignments of colors in $S = \{abbc, abca, abcc\}$ to the boundary are feasible for G.*

We can classify 4-sided near-triangulations into 3 types depending on which two of the three assignments in S are feasible for them. Note that for some near-triangulations all 3 assignments may be feasible, in which case it can be assumed to be any type. The three types correspond to the simplest 4-sided near-triangulations, two with no internal vertex and a chord v_1v_3 or v_2v_4, and the other with exactly one internal vertex adjacent to all four vertices on the boundary. We show that every 4-sided near-triangulation is of one of these types. We will say a near-triangulation is of type $\mathcal{T}_1$ if the assignments $abbc, abca$ are feasible for it. It is of type $\mathcal{T}_2$ if the assignments $abca, abcc$ are feasible for it and of type $\mathcal{T}_3$ if $abbc, abcc$ are feasible assignments.

Proof: Suppose the theorem and/or the lemma is false and consider a counterexample G to either with the minimum number of vertices.

We first show that G must be simple and cannot contain a separating triangle, that is, a triangle that has vertices of G in its interior as well as exterior.

Suppose G has multiple edges, and let e_1, e_2 be two edges in G with the same endpoints p, q. Let C be the 2-cycle containing the two edges. Since every face is 3 sided, there must be vertices in the interior as well as exterior of C and at least one of the edges, say e_1, is an internal edge. Let G' be the graph obtained from G by deleting all the vertices and edges in the interior of C and the edge e_1. Then G' is also a triangulation or a near-triangulation with the same boundary but fewer vertices than G. We show that any assignment of colors to the boundary that is feasible for G' is also feasible for G. Suppose f' is a special 3-coloring of G'. Let G'' be the triangulation obtained from G by deleting the vertices and edges in the exterior of the cycle C and the edge e_1, and let the boundary of G'' be the boundary of a face that contains e_2. Relabel the vertex p as v_1 and q as v_2 in G''. Suppose $\{f'(p)\} \cup \{f'(q)\} \neq \{a, b\}$. Without loss of generality, assume $b \notin \{f'(p)\} \cup \{f'(q)\}$. Lemma 1 implies, after swapping colors if necessary, that there exists a non-monochromatic 2-coloring f'' of G'' with colors $\{a, c\}$ such that $f''(p) = f'(p)$ and $f''(q) = f'(q)$. If $\{f'(p)\} \cup \{f'(q)\} = \{a, b\}$, then we may assume $p = v_1$ and $q = v_2$. Again by the minimality of G, G'' has a special 3-coloring f'' such that $f''(p) = f'(p)$ and $f''(q) = f'(q)$. In both cases, setting $f(v) = f'(v)$ for all $v \in V(G')$ and $f(v) = f''(v)$ for all $v \in V(G'')$ gives a special 3-coloring of G with the same assignment of colors to the boundary as in G'. This contradicts the assumption that G is a counterexample.

Suppose G is simple but contains a separating triangle C with vertices p, q, r. Again, let G' be the triangulation obtained from G by deleting the vertices and edges in the interior of C and G'' the triangulation obtained from G by deleting the vertices and edges in the exterior of C. Suppose f' is any special 3-coloring of G' with some assignment of colors to the boundary. Since C is an internal face in G', $|\{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\}| \geq 2$. If $|\{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\}| = 2$, then $\{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\} \neq \{a, b\}$, since we may assume $r \notin \{v_1, v_2\}$ and either $f'(r) \neq f'(p)$ or $f'(r) \neq f'(q)$. Again, we may assume without loss of generality that the color b is not assigned to any of the vertices p, q, r in f'. Lemma 1 then implies, again after swapping colors and relabeling vertices if necessary, that there exists a non-monochromatic 2-coloring f'' of G'' with colors $\{a, c\}$ such that $f''(v) = f'(v)$ for all $v \in \{p, q, r\}$. If $|\{f'(p)\} \cup \{f'(q)\} \cup \{f'(r)\}| = 3$, we may assume without loss of generality that $p = v_1$, $q = v_2$, $f'(p) = a$, $f'(q) = b$ and $f'(r) = c$. Considering v_1, v_2, r to be the boundary of G'', by the minimality of G, there is a special 3-coloring f'' of G'' such that $f''(v) = f'(v)$ for all $v \in \{p, q, r\}$. In either case, setting $f(v) = f'(v)$ for all $v \in V(G')$ and $f(v) = f''(v)$ for all $v \in V(G'')$ gives a special 3-coloring of G with the same assignment of colors to the boundary as in G'. Again, this contradicts the assumption that

G is a counterexample.

We now assume G is simple and has no separating triangles and consider cases depending on whether G is a triangulation or a near-triangulation with a 4-sided boundary.

Case 1: Suppose G is a triangulation. If G contains a vertex of degree 2, since there are no multiple edges, G must be K_3, in which case, the assignment abc of colors to the boundary is a special 3-coloring of G. If G contains a vertex of degree 3, since there are no separating triangles, G must be K_4. However, in this case, assigning color c to the internal vertex and abc to the boundary gives a special 3-coloring of G. So we may assume every vertex in G has degree at least 4.

We next show that v_3 must have degree exactly 4. Suppose not and let v_1, v_4, v_5, v_6 be the neighbors of v_3 in circular order, such that v_1, v_3, v_4, and v_3, v_4, v_5 are internal faces in G. In any special 3-coloring of G with abc assigned to the boundary, v_3 must be assigned color c. The vertex v_5 is not adjacent to either v_1 or v_2, otherwise there is a separating triangle in G, separating v_4 and v_6. Let G' be the triangulation obtained from G by flipping the edge v_3v_4, replacing it by v_1v_5, and then contracting v_1v_5 to the vertex v_1. Then G' has the same boundary as G and no self-loops, hence by the minimality of G, it has a special 3-coloring f' with the colors abc assigned to the boundary. Define $f(v_5) = f'(v_1) = a$ and $f(v) = f'(v)$ for all $v \in V(G')$. We show that f is a special 3-coloring of G. All faces in G except v_1, v_3, v_4 and v_3, v_4, v_5 correspond to faces in G', with vertex v_5 replaced by v_1 if it appears on the boundary of the face. Since $f(v_5) = f'(v_1)$ and $f(v) = f'(v)$ for all other vertices, these faces are non-monochromatic in f. Since $f(v_3) = f'(v_3) = c$ and $f(v_5) = f(v_1) = f'(v_1) = a$, the other two faces are also non-monochromatic. The only pairs of adjacent vertices in G that are not adjacent in G' are v_3, v_4 and those that include v_5 and one of its neighbors. Since v_3 is colored c, it cannot form an ab-pair with any other vertex. Any neighbor of v_5 in G is a neighbor of v_1 in G'. The only neighbor of v_1 that is colored b is v_2, which is not a neighbor of v_5. Therefore there are no ab-pairs in f apart from v_1, v_2 and f is a special 3-coloring of G, a contradiction. Note that this argument fails if the degree of v_3 is exactly 4, since v_5 is now adjacent to v_2.

We now assume the degree of v_3 is exactly 4 and as before, let v_1, v_4, v_5, v_2 be its four neighbors in circular order. Let G' be the near-triangulation obtained from G by deleting the vertex v_3, having the boundary v_1, v_2, v_5, v_4 in circular order. The minimality of G implies G' is of one of types $\mathcal{T}_1$, $\mathcal{T}_2$, or $\mathcal{T}_3$. In all 3 cases, there exists a special 3-coloring f' of G' with the boundary assigned colors $abbc$ or $abca$. In either case, setting $f(v_3) = c$ and $f(v) = f'(v)$ for all $v \in V(G')$ gives a special 3-coloring of G, with colors abc assigned to the boundary. This contradicts the assumption that G is a counterexample.

Case 2: Suppose G is a 4-sided near-triangulation. Suppose G has a chord, in which case there are no internal vertices in G. If the chord is v_1v_3, the assignments $abca$ and $abcc$ are special 3-colorings of G, but $abbc$ is not since the vertex v_1 with color a is adjacent to the vertex v_3 of color b. In this case, G is of type $\mathcal{T}_2$. If the chord is v_2v_4 then the assignments $abbc$ and $abcc$ are special 3-colorings and G is of type $\mathcal{T}_3$.

We may assume G has no chords. Let v_5 be the internal vertex in G such that v_1, v_2, v_5 is the internal face in G containing the edge v_1v_2. If v_5 is adjacent to both v_3 and v_4, then there are no other vertices in G. Assigning color c to v_5 and either colors $abbc$ or $abca$ to the boundary, gives a special 3-coloring of G. Thus G is of type $\mathcal{T}_1$.

Suppose v_5 is adjacent to v_3 but not to v_4. Let G' be the triangulation obtained from G

by deleting the vertex v_2 and adding the edge v_1v_3 in the exterior of the cycle v_1, v_5, v_3, v_4. Then G' is a triangulation with boundary v_1, v_3, v_5 and by the minimality of G, G' has a special 3-coloring f' with colors abc assigned to the boundary. Since v_4 is adjacent to both v_1 and v_3, we must have $f'(v_4) = c$. Setting $f(v_2) = b$ and $f(v) = f'(v)$ for all $v \in V(G')$ gives a special 3-coloring of G with colors $abbc$ assigned to the boundary. Similarly, Lemma 1 implies that G' has a non-monochromatic 2-coloring f'' with colors acc assigned to the boundary. In this case, v_4 may be colored either a or c. Setting $f(v_2) = b$ and $f(v) = f''(v)$ for all $v \in V(G')$, gives a special 3-coloring of G with either $abca$ or $abcc$ assigned to the boundary. Then G is of type $\mathcal{T}_1$ or $\mathcal{T}_3$, depending on the color of v_4 in f''. A symmetrical argument holds if v_5 is adjacent to v_4 but not to v_3.

We may now assume v_5 is not adjacent to either v_3 or v_4. The degree of v_5 is at least four, since G has no separating triangles. Again, we show that the degree must be exactly four, using essentially the same argument as in the case of triangulations. Note that in any special 3-coloring of G, v_5 must be assigned color c. Suppose the degree of v_5 is at least five and let v_2, v_6, v_7, v_8 be vertices adjacent to v_5 in circular order, such that v_2, v_5, v_6 and v_5, v_6, v_7 are internal faces in G. The vertex v_7 cannot be adjacent to either v_1 or v_2, otherwise G has a separating triangle. Let G' be obtained from G by flipping the edge v_5v_6, replacing it by v_2v_7 and then contracting the edge v_2v_7 to the vertex v_2. Then G' has the same boundary as G and fewer vertices, and by the minimality of G, is of one of the three types. As argued in the case of triangulations, any special 3-coloring of G' with an assignment of colors from S to the boundary, can be extended to a special 3-coloring of G with the same assignment to the boundary, by assigning the color of v_2 (b) to v_7. Therefore G is of the same type as G', a contradiction.

Suppose the degree of v_5 is exactly 4 and let v_1, v_2, v_6, v_7 be its neighbors in circular order. The previous argument cannot be used now as v_7 is adjacent to v_1 and v_1, v_7 would become an ab-pair in the coloring f of G.

Suppose there exists a cycle C of length four v_1, v_2, p, q in G such that $\{p, q\} \not\subseteq \{v_5, v_6, v_7\}$ and $\{p, q\} \neq \{v_3, v_4\}$. Let G' be the near-triangulation obtained from G by deleting the vertices in the exterior of the cycle C, with the boundary v_1, v_2, p, q. Since at least one of v_3, v_4 must be in the exterior of C, G' has fewer vertices than G. The minimality of G implies G' is of one of the three types. We replace the vertices in the interior of C by an equivalent smaller subgraph of the same type. Note that at least two of the vertices v_5, v_6, v_7 must be contained in the interior of C. Let H be the graph obtained from G by deleting the vertices in the interior of C. If G' is of type $\mathcal{T}_1$, add a new vertex r to H in the interior of the cycle C and join it to all four vertices v_1, v_2, p, q and call the resulting graph G''. If G' is of type $\mathcal{T}_2$, add the edge v_1p in the interior of C to construct G'' from H. If G' is of type $\mathcal{T}_3$, add the edge v_2q in the interior of C to construct G'' from H. In all cases, G'' is a near-triangulation with the same boundary but fewer vertices than G, and by the minimality of G, is of one of the three types. We claim that G is of the same type as G''. Let f'' be any special 3-coloring of G''. If G'' was constructed by adding the vertex r in the interior of C, since r is adjacent to v_1 and v_2, we must have $f''(r) = c$. This implies that either $f''(p) = b$ and $f''(q) = c$ or $f''(p) = c$ and $f''(q) = a$. Since in this case G' was of type $\mathcal{T}_1$, there exists a special 3-coloring f' of G' such that $f'(p) = f''(p)$ and $f'(q) = f''(q)$. Similarly, in the other cases, it can be argued that the possible values of $f''(p)$ and $f''(q)$ are such that there exists a special 3-coloring f' of G' such that $f'(p) = f''(p)$ and $f'(q) = f''(q)$.

In all cases, we have $f'(v_1) = f''(v_1) = a$ and $f'(v_2) = f''(v_2) = b$. Defining $f(v) = f'(v)$ for all $v \in V(G')$ and $f(v) = f''(v)$ for all $v \in V(G'')$ gives a special 3-coloring of G with the same assignment of colors to the boundary as in the coloring of G''. This implies G is of the same type as G'', a contradiction.

We may now suppose there is no such cycle C in G. This implies that v_7 is not adjacent to v_3 and v_6 is not adjacent to v_4. Suppose v_7 is adjacent to v_4 and v_6 is adjacent to v_3. We claim that G is of type $\mathcal{T}_1$ in this case. Let G' be the triangulation obtained from G by deleting the vertices v_1, v_2, v_5 and adding the edge v_3v_7 in the exterior of the cycle v_3, v_4, v_7, v_6. Let the boundary of G' be v_7, v_3, v_4. The minimality of G implies that G' has a special 3-coloring f' with colors abc assigned to the boundary. Since v_6 is adjacent to both v_3 and v_7, we have $f'(v_6) = c$. Setting $f(v_1) = a$, $f(v_2) = b$, $f(v_5) = c$ and $f(v) = f'(v)$ for all $v \in V(G')$ gives a special 3-coloring of G with colors $abbc$ assigned to the boundary. Similarly, by deleting the vertices v_1, v_2, v_5 and adding the edge v_4v_6 to construct a triangulation G'' with boundary v_4, v_6, v_3, we can find a special 3-coloring of G with colors $abca$ assigned to the boundary. Thus G is of type $\mathcal{T}_1$, a contradiction.

Suppose, without loss of generality, that v_6 is not adjacent to v_3. The argument in the other case is symmetrical, after relabeling the vertices and swapping the colors a and b. Let $w \neq v_5$ be the internal vertex in G such that v_2, v_6, w is the other face in G containing the edge v_2v_6. We show that we can find special 3-colorings of G with the additional condition that w is colored c. Let $v_2 = w_1, w_2, \ldots, w_d = v_6$ be the vertices adjacent to w in circular order, starting with v_2 and ending with v_6, where $d \geq 4$ is the degree of w. Note that if w is adjacent to v_3, we must have $w_2 = v_3$, otherwise G has a separating triangle. The vertex v_1 cannot be adjacent to any vertex w_i for $i > 1$, otherwise v_1, v_2, w, w_i is a 4-cycle in G. This implies v_4, v_7 are not adjacent to w. Also, no w_i can be adjacent to a w_j for $1 < |i - j| < d - 1$, otherwise G has a separating triangle.

Construct a near-triangulation G' from G as follows. First flip all the edges ww_{2i} and replace them by the edges $w_{2i-1}w_{2i+1}$ for $1 \leq i < \lceil d/2 \rceil$. Then contract all the edges $w_{2i-1}w_{2i+1}$ for $1 \leq i < \lceil d/2 \rceil$ to the vertex $w_1 = v_2$. If d is even, the vertex w will have degree 2 in the resulting graph with v_2 and v_6 as its two neighbors. There will be two edges between v_6 and v_2, corresponding to the edges $v_2v_6 = w_1w_d$ and $w_{d-1}w_d$ in G, and w is the only vertex contained in the interior of the 2-cycle formed by the two edges. There will be no self-loops in the resulting graph. If d is odd, w will have degree 1 with v_2 as its only neighbor. There will be a self-loop at v_2 corresponding to the edge w_1w_d in G whose interior will contain the vertex w and the edge v_2w. The other face containing this loop will contain two edges between v_2 and v_5, corresponding to the original edges v_2v_5 and v_5v_6 in G. In this case, we delete the vertex w, the self loop at v_2 and one of the two edges between v_2 and v_5 that are in the same face as the self-loop at v_2. Let G' be the resulting graph.

Note that G' has the same boundary as G as neither v_3 nor v_4 will be merged with v_2. There are no self-loops in G' and since at least one edge will be contracted, G' has fewer vertices than G. The minimality of G, implies that G' is of one of the three types. We show that G is of the same type as G'. Consider any special 3-coloring f' of G' with some assignment of colors in $\{abbc, abca, abcc\}$ to the boundary. In the case d is even, since the two faces whose boundary contains w both have v_2 and v_6 as the other two vertices, and $f'(v_2) = b$, we can set $f'(w) = c$, irrespective of the color of v_6. Now we claim that setting $f(w_{2i+1}) = b$ for $1 \leq i < \lceil d/2 \rceil$, $f(w) = c$ and $f(v) = f'(v)$ for all $v \in V(G')$ gives a special 3-coloring of G with the same assignment of colors to the boundary as in f'. Any face in G that does not include w on its boundary, except possibly the face v_2, v_5, v_6 when d is odd, corresponds to a face in G' with the vertex w_{2i+1} replaced by $w_1 = v_2$ if w_{2i+1} is on its boundary for some $1 \leq i < \lceil d/2 \rceil$. Since all vertices w_{2i+1} have the same color b as w_1, these are non-monochromatic in f as well. The faces in G that include w are non-monochromatic since $f(w) = c$ and $f(w_{2i-1}) = b$ for $1 \leq i \leq \lceil d/2 \rceil$. The face v_2, v_5, v_6 is non-monochromatic since $f(v_5)$ must be c. There cannot be any ab-pairs in f apart from v_1, v_2, since the only vertex of color a adjacent to v_2 in G' is v_1, but none of the vertices w_{2i+1} for $1 \leq i < \lceil d/2 \rceil$ is adjacent to v_1. Thus G has a special 3-coloring with the same assignment of colors to the boundary as in G' and hence is of the same type as G'.

This completes the proof. $\square$

4 Disconnected 2-factors

In this section, we consider disconnected 2-factors in 2-edge-connected planar cubic graphs. This is equivalent to finding separating perfect matchings in the graph. While the existence of these in all 2-edge-connected planar cubic graphs with at least six vertices was shown in [2], here we consider their existence with the additional restriction that they should include a specified edge. We give a constructive characterization of planar cubic 2-edge-connected graphs that contain an edge such that every 2-factor containing the edge is a Hamiltonian cycle. We also characterize 2-edge-connected planar cubic graphs that have an edge that is not contained in a separating perfect matching.

We first show using Theorem 1 that cyclically 4-edge-connected planar cubic graphs except K_2^3 and K_4 do not contain any such edges.

Theorem 3 *Let G be a 2-edge-connected planar cubic graph and let uv be any edge in G such that there is no edge parallel to it and $G - \{u, v\}$ is 2-edge-connected. Then there exists a separating perfect matching in G including the edge uv.*

Proof: Consider any plane embedding of G. Since there is no edge parallel to uv, both u, v must have 2 neighbors each not in $\{u, v\}$. Let u_1, u_2 be the other neighbors of u and v_1, v_2 the other neighbors of v such that u_1, u, v, v_1 are consecutive vertices on the boundary of a face of G, as are u_2, u, v, v_2. The vertices u_1, u_2, v_1, v_2 must all be distinct otherwise there is a bridge in $G - \{u, v\}$. Let G' be the planar cubic graph obtained from $G - \{u, v\}$ by adding two new vertices u', v' adjacent to each other, joining u' to u_1, v_1 and v' to u_2, v_2. Since $G - \{u, v\}$ is 2-edge-connected, $u'v'$ is not contained in a 2-edge-cut in G'. Theorem 1 implies there is a 2-factor in G' such that $u'v'$ is a chord of some cycle C in the 2-factor. The cycle C must contain the edges u_1u', $u'v_1$, u_2v' and $v'v_2$. Then planarity implies $C - \{u', v'\}$ is the disjoint union of a path from u_1 to u_2 and a path from v_1 to v_2. Adding the edges u_1u, u_2u, v_1v and v_2v to these two paths gives two disjoint cycles in G which include all vertices in C except u', v'. These two cycles, along with the cycles other than C in the 2-factor of G' (if any), give a disconnected 2-factor in G not containing the edge uv. The complement of this is a separating perfect matching in G containing uv. $\square$

An immediate consequence of Theorem 3 is that every edge in a cyclically 4-edge-connected planar cubic graph, other than K_2^3 and K_4, is contained in a separating perfect matching. Any edge uv in such a graph satisfies the condition $G-\{u,v\}$ is 2-edge-connected. This also implies that every edge in such a graph is contained in a disconnected 2-factor. The result in [2] that every 2-edge-connected planar cubic graph except K_2^3 and K_4 has a disconnected 2-factor also follows easily from this. If the graph is not cyclically 4-edge-connected, it must have a cyclic cut of size at most 3. If it has a 2-edge-cut then any perfect matching containing an edge in the cut must also contain the other edge and is a separating perfect matching. Petersen's theorem implies there exists such a matching. On the other hand, if the graph G is 3-edge-connected, any edge uv contained in a cyclic cut of size 3 must satisfy $G-\{u,v\}$ is 2-edge-connected, otherwise there is a 2-edge-cut in G.

We next consider 3-edge-connected planar cubic graphs and characterize those that have an edge that is not contained in a disconnected 2-factor and also those that have an edge not contained in a separating perfect matching. Let H_n for $n \geq 1$ denote the graph obtained from a cycle $v_0, v_1, \ldots, v_{2n-1}$ of length $2n$ by adding the edges v_0v_n and v_iv_{2n-i} for $1 \leq i < n$. Note that H_2 is K_4 and H_3 is the prism $K_3 \times K_2$.

Theorem 4 *Let G be a 3-edge-connected planar cubic graph with an edge uv such that every 2-factor of G that contains uv is a Hamiltonian cycle. Then G is H_n for some $n \geq 1$ and uv is the edge v_0v_n.*

Proof: The proof is by induction on the number of vertices. If G has only two vertices, it must be K_2^3 and the result holds. Suppose G has $2n$ vertices for some $n > 1$. Let u_1, u_2 be the neighbors of u other than v. Theorem 3 implies $G-\{u,u_1\}$ as well as $G-\{u,u_2\}$ contains a bridge, otherwise there is a separating perfect matching containing uu_1 or uu_2, whose complement is a disconnected 2-factor in G containing the edge uv, a contradiction. Let pq be a bridge in $G-\{u,u_1\}$, let C_1, C_2 be the components of $(G-\{u,u_1\})-pq$ and assume without loss of generality that p is in C_1 and q in C_2. Since G is 3-edge-connected, u_2 and v belong to different components, and without loss of generality, u_2 is in C_1. Note that u_1 also has one neighbor in C_1 and one in C_2. If the component C_1 is non-trivial, it must contain at least 3 vertices and must be 2-connected, otherwise G has a 2-edge-cut. The same holds for C_2. This implies that if C_1 is not trivial, we can find a 2-edge-connected subgraph in $G-\{u,u_2\}$ containing both v and u_1. This is obtained by adding a path to C_2 that goes through u_1 and pq, and uses a path in $C_1-\{u_2\}$ between p and the vertex in C_1 that is adjacent to u_1. This contradicts the fact that there must be a bridge that separates v and u_1 in $G-\{u,u_2\}$. Therefore C_1 must be trivial and contain only the vertex $p=u_2$. This implies u, u_1, u_2 form a triangle in G. Let G' be obtained from G by contracting the triangle u, u_1, u_2 to the vertex u. If uv is a contained in a disconnected 2-factor in G', then it is also contained in a disconnected 2-factor in G, obtained by adding appropriate edges from the contracted triangle. Otherwise G' must be H_{n-1} and uv must be the edge v_0v_{n-1}. Then the planarity of G implies that G must be H_n and uv the edge v_0v_n. □

Theorem 3 implies that if uv is an edge in a 3-edge-connected planar cubic graph other than K_2^3 that is not contained in a separating perfect matching, then $G-\{u,v\}$ has a bridge say pq. Let C_1, C_2 be the components of $(G-\{u,v\})-pq$ and let G_1 and G_2 be the graphs obtained from G by contracting the vertices $V(C_2)\cup\{u,v\}$ and $V(C_1)\cup\{u,v\}$ to the vertices x and y, respectively. If there is a separating perfect matching in either G_1 containing the edge px or one in G_2 containing the edge qy, then we get a separating perfect matching in G including uv, a contradiction. Therefore both G_1 and G_2 are 3-edge-connected planar cubic graphs and the edges px and qy are not contained in a separating perfect matching in G_1 and G_2, respectively. The converse of this is also true and any graph G constructed from G_1 and G_2 in this way has the edge uv not contained in a separating perfect matching. The base case is if G is K_2^3 and any other 3-edge-connected planar cubic graph with an edge not contained in a separating perfect matching can be obtained from K_2^3 using this operation.

Finally, we consider the 2-edge-connected case. Suppose G is a 2-edge-connected planar cubic graph with an edge uv that is not contained in a disconnected 2-factor. If uv is contained in a 2-edge-cut, let pq be the other edge in the cut. Let C_1, C_2 be the components of $G - \{uv, pq\}$ and without loss of generality, assume $u, p \in V(C_1)$ and $v, q \in V(C_2)$. Let G_1, G_2 be the graphs obtained from C_1, C_2 by adding the edges up, vq, respectively. If there is a disconnected 2-factor in either G_1 that contains the edge up, or a disconnected 2-factor in G_2 that contains vq, then we get a disconnected 2-factor in G containing uv. Therefore both G_1, G_2 are 2-edge-connected planar cubic graphs with edges up, vq that are not contained in a disconnected 2-factor, respectively. The converse of this also follows easily, and any graph G constructed in this way, has the edge uv not contained in a disconnected 2-factor.

Suppose the edge uv is not contained in a 2-edge-cut. If G is 3-edge-connected, it follows from Theorem 4 that G is H_n for some $n \geq 1$ and uv is the edge v_0v_n. Suppose there exists a 2-edge-cut $\{p_1q_1, p_2q_2\}$ in G such that p_1, p_2 are vertices in the component C_1 of $G - \{p_1q_1, p_2q_2\}$ that contains the edge uv. Let C_2 be the other component. Choose such a cut such that the size of C_2 is as large as possible. Let G_1, G_2 be the graphs obtained from C_1, C_2 by adding the edges p_1p_2, q_1q_2, respectively. Then uv is not contained in a 2-edge-cut in G_1, otherwise we get a 2-edge-cut in G containing uv. Also, if there is a 2-edge-cut in G_1, the vertices u, v, p_1, p_2 must be contained in the same component of the cut, otherwise it contradicts the assumption that the size of C_2 was as large as possible. Repeat this process of 2-edge-cut reduction on G_1 until we are left with a 3-edge-connected graph G_1 that contains the edge uv. Thus the graph G may be viewed as being obtained from a 3-edge-connected planar cubic graph G_1 by replacing some of the edges by the graphs G_2.

If the final graph G_1 has a disconnected 2-factor containing the edge uv, we can find one in G as well. If p_1p_2 is any added edge in the graph G_1, if the 2-factor does not contain the edge p_1p_2, adding a 2-factor in G_2 not containing the edge q_1q_2 gives a disconnected 2-factor in G. If p_1p_2 is included in the 2-factor, take a 2-factor in G_2 containing the edge q_1q_2. Replacing the edges p_1p_2, q_1q_2 by the edges p_1q_1, p_2q_2 gives a disconnected 2-factor in G. Theorem 4 then implies the final graph G_1 must be H_n for some $n \geq 1$. The same argument holds if any of the graphs G_2 that replace an edge in G_1 has a disconnected 2-factor containing the edge q_1q_2. Take any 2-factor in G_1 containing the edge uv. If the edge p_1p_2 is not in the 2-factor, we add any 2-factor in G_2 not containing q_1q_2, otherwise add a disconnected 2-factor containing q_1q_2 and replace the edges p_1p_2, q_1q_2 by p_1q_1, p_2q_2. Finally, if G_1 has a 2-factor containing uv but not containing any one of the added edges p_1p_2, we get a disconnected 2-factor by adding a 2-factor in G_2 not containing q_1q_2. Therefore the only case when we cannot get a disconnected 2-factor is if G_1 is H_n for some $n \geq 1$, the edge uv is v_0v_n and any 2-factor in H_n that contains v_0v_n also includes the replaced edges p_1p_2. There are only two perfect matchings in H_n that do not contain the edge v_0v_n, depending on whether it includes v_0v_1 or v_0v_{2n-1}. In either case, the matchings cannot include the edges v_iv_{2n-i} for $1 \leq i < n$ and any 2-factor that includes v_0v_n must include these edges. These are precisely the edges that can be replaced by the graphs G_2, and G must have been obtained from some H_n for $n \geq 1$ by these operations. The edge uv is still the edge $v_0 v_n$ of H_n.

This describes the structure of all 2-edge-connected planar cubic graphs that have an edge uv such that every 2-factor containing the edge is a Hamiltonian cycle. We now consider the same for all such graphs that have an edge uv that is not contained in any separating perfect matching.

Let G be a 2-edge-connected planar cubic graph and uv an edge in G that is not contained in any separating perfect matching. Suppose there exists another edge between u and v. Then either G is K_2^3 or u, v have distinct neighbors u_1, v_1, respectively, not in $\{u, v\}$. Assume G is not K_2^3 and let G' be obtained from $G - \{u, v\}$ by adding the edge $u_1 v_1$. Then G' is also a 2-edge-connected planar cubic graph and it is easy to see that G has a separating perfect matching including uv iff G' has a disconnected 2-factor containing the added edge $u_1 v_1$. This gives a way of constructing graphs having an edge not contained in a separating perfect matching from graphs with an edge not contained in a disconnected 2-factor.

If uv is contained in a 2-edge-cut then any perfect matching containing uv must contain the other edge in the 2-edge-cut, and any such perfect matching is a separating perfect matching. Petersen's theorem implies there exists such a matching, a contradiction.

Suppose $G - \{u, v\}$ is disconnected and let C_1, C_2 be the components of $G - \{u, v\}$. Let u_1, u_2 be the neighbors of u and v_1, v_2 the neighbors of v in C_1, C_2, respectively. Let G_1, G_2 be the graphs obtained from C_1, C_2 by adding the edges $u_1 v_1$, $u_2 v_2$, respectively. Now G has a separating perfect matching containing the edge uv iff either G_1 or G_2 has a separating perfect matching not including $u_1 v_1$ or $u_2 v_2$, respectively. In other words, there is no disconnected 2-factor in G_1 or G_2 containing the edges $u_1 v_1$ or $u_2 v_2$, respectively. The converse also follows similarly. This gives another way of constructing a graph having an edge not contained in a separating perfect matching from 2 smaller graphs having an edge not contained in a disconnected 2-factor.

If $G - \{u, v\}$ is connected but has a bridge, it follows from the same argument as in the 3-edge-connected case that G can be constructed from two smaller graphs having an edge not contained in a separating perfect matching. Finally, if $G - \{u, v\}$ is 2-edge-connected, since there is no edge parallel to uv, it follows from Theorem 3 that there exists a separating perfect matching including uv, a contradiction.

This covers all possibilities and gives a constructive characterization of 2-edge-connected planar cubic graphs that have an edge not contained in a separating perfect matching.

5 Extension

It seems possible that the results in this paper can be extended further. In particular, it appears to be true that for any two edges in a 2-edge-connected planar cubic graph, there exists a 2-factor in the graph such that the two edges are contained in the same cycle of the 2-factor. If the two edges are on the boundary of the same face, we can subdivide the two edges and add an edge joining the two degree 2 vertices in the interior of the face. The added edge is not contained in a 2-edge-cut in the resulting 2-edge-connected planar cubic graph, and Theorem 1 implies it is a chord of some cycle in a 2-factor of the graph. This gives a 2-factor in the original graph such that both edges are contained in the same cycle.

It is also possible that the same property holds for all connected cubic bipartite graphs. Here we prove it for connected planar cubic bipartite graphs.

Theorem 5 *Let G be a connected planar cubic bipartite graph and let e_1, e_2 be any two edges in G. There exists a 2-factor in G such that both the edges are contained in the same cycle of the 2-factor.*

Proof: We prove by induction on the number of vertices. Note that since G is cubic and bipartite, it cannot have a bridge. Suppose G has a 2-edge-cut $\{p_1q_1, p_2q_2\}$. Let C_1, C_2 be the components of $G - \{p_1q_1, p_2q_2\}$ and assume $p_1, p_2 \in V(C_1)$ and $q_1, q_2 \in V(C_2)$. Let G_1, G_2 be the graphs obtained by adding the edges p_1p_2, q_1q_2 to C_1, C_2 respectively. We will call the edge p_1p_2 as e_1 if e_1 is in the 2-edge-cut and similarly call q_1q_2 as e_2 if e_2 is in the 2-edge-cut. If e_1, e_2 are now contained in the same graph, say G_1, by induction, there exists a 2-factor F_1 in G_1 such that e_1, e_2 are in the same cycle of F_1. If F_1 does not contain the edge p_1p_2, let F_2 be any 2-factor in G_2 not containing the edge q_1q_2. Then $F_1 \cup F_2$ is a 2-factor in G such that e_1 and e_2 are contained in the same cycle of the 2-factor. If F_1 contains p_1p_2 then let F_2 be a 2-factor in G_2 containing q_1q_2. Then $F = (F_1 \setminus \{p_1p_2\}) \cup (F_2 \setminus \{q_1q_2\}) \cup \{p_1q_1, p_2q_2\}$ is a 2-factor in G such that e_1, e_2 are contained in the same cycle of F. Finally, suppose e_1, e_2 are in different graphs, say e_1 is in G_1 and e_2 in G_2. By induction, there is a 2-factor F_1 in G_1 such that e_1 and p_1p_2 are in the same cycle of F_1. Similarly, there is a 2-factor F_2 in G_2 such that e_2 and q_1q_2 are in the same cycle of F_2. Then $F = (F_1 \setminus \{p_1p_2\}) \cup (F_2 \setminus \{q_1q_2\}) \cup \{p_1q_1, p_2q_2\}$ is a 2-factor in G such that e_1, e_2 are contained in the same cycle of F.

We may now assume G is 3-edge-connected. If G is K_2^3 the result is obviously true. If G has at least 4 vertices then there are no multiple edges and since G is bipartite, every face in a plane embedding of G has even size at least four. Euler's formula implies G has at least 6 faces of size 4, and hence at least two such that they do not contain any of the edges e_1, e_2 on their boundary. Let v_1, v_2, v_3, v_4 be the vertices on the boundary of such a face and let u_i be the neighbor of v_i not on the boundary of the face, for $1 \leq i \leq 4$. The u_i must be distinct vertices since G is planar and bipartite. If e_1 is one of the edges u_iv_i, we may assume without loss of generality, it is u_1v_1. If e_2 is also incident with some vertex v_i, we may assume it is one of u_2v_2 or u_3v_3. Let G' be the graph obtained from G by deleting the vertices v_1, v_2, v_3, v_4 and adding the edges u_1u_4 and u_2u_3. Then G' is a cubic bipartite graph and must be connected, otherwise G has a 2-edge-cut. Label the added edge u_1u_4 as e_1 if e_1 was the edge u_1v_1. Similarly, label u_2u_3 as e_2 if either u_2v_2 or u_3v_3 was e_2. By induction, G' has a 2-factor F' such that e_1 and e_2 are in the same cycle of F'. If F' does not contain any of the added edges, then adding the edges in the 4-cycle v_1, v_2, v_3, v_4 to F' gives a 2-factor in G such that e_1, e_2 are in the same cycle. If F' contains u_1u_4 but not u_2u_3, then $(F' \setminus \{u_1u_4\}) \cup \{u_1v_1, v_1v_2, v_2v_3, v_3v_4, u_4v_4\}$ is a 2-factor in G such that e_1, e_2 are contained in the same cycle. The same argument holds if F' contains u_2u_3 but not u_1u_4. If F' contains both, then $(F' \setminus \{u_1u_4, u_2u_3\}) \cup \{u_1v_1, u_2v_2, u_3v_3, u_4v_4, v_1v_4, v_2v_3\}$ is a 2-factor in G such that e_1, e_2 are in the same cycle of the 2-factor. □

Another possible extension is to consider 2-factors including two specified edges, and characterize those 2-edge-connected planar cubic graphs that have a pair of edges such that every 2-factor containing both of them is a Hamiltonian cycle.

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