Edge-disjoint paths with three terminals

Ajit A. Diwan and Namrata P. Tholiya
Department of Computer Science and Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076,India.
email: aad@cse.iitb.ac.in

August 25, 2008

Abstract

We prove that in every simple graph G with minimum degree $d \geq 2$, there are edges $\{uv, vw\}$ such that G contains $\lfloor 3d/2 \rfloor$ edge-disjoint $\{u, v, w\}$-paths. If d is even, the paths can be chosen such that each pair of vertices in $\{u, v, w\}$ is joined by $d/2$ paths. If d is odd, any specified pair of vertices in $\{u, v, w\}$ can be joined by $(d+1)/2$ of these paths and the other two pairs by $(d-1)/2$ paths. This is not true for multigraphs with minimum degree d in general. We show that in every multigraph H of order at least k and minimum degree d, there is a set A of k vertices, such that H contains a collection of $\lfloor dk/2 \rfloor$ edge-disjoint A-paths and A-cycles, where an A-cycle is a cycle containing exactly one vertex in A.

1 Introduction

A well-known theorem of Mader [3, 4] gives a min-max relation for the maximum number of edge-disjoint (or vertex-disjoint) paths in a graph, whose endpoints are contained in a given set A of vertices, called terminals, and whose internal vertices are not in A. Such paths are called A-paths. A short proof of Mader's theorem is given in [5]. We consider a variation of this problem, where only the number k of terminals is specified, and a set A of k terminals is to be chosen to maximize the number of edge-disjoint A-paths.

The motivation for this problem is another theorem of Mader [1] that every multigraph with minimum degree $d \geq 1$ contains an edge uv such that there are d edge-disjoint paths between u and v. Subsequently, Mader [2] showed that the paths can in fact be chosen to be internally vertex-disjoint, if the graph is simple. We consider the analogous problem for A-paths, when $|A|$ is specified.

In particular, we consider the case when the number of terminals is three. We show that in every simple graph G with minimum degree $d \geq 2$, there are two edges $\{uv, vw\}$ such that there are $\lfloor 3d/2 \rfloor$ edge-disjoint $\{u, v, w\}$-paths in G. If d is even, the paths can be chosen such that there are $d/2$ paths between each pair of vertices in $\{u, v, w\}$. If d is odd, the paths can be chosen such that any specified pair of vertices in $\{u, v, w\}$ is joined by $(d+1)/2$ paths and the other two pairs by $(d-1)/2$ paths. However, unlike the two terminal case, this does not hold for multigraphs.

If G is a d-regular multigraph, the maximum possible number of edge-disjoint paths with k terminals is $\lfloor dk/2 \rfloor$, since every path contributes two to the sum of degrees of terminal vertices. Mader's theorem shows that this bound is achieved in every multigraph with minimum degree d, when $k = 2$. However, this is not the case when $k > 2$.

We show that in every multigraph G of order at least k and minimum degree d, there is a set A of k vertices, such that G contains a collection of $\lfloor dk/2 \rfloor$ edge-disjoint A-paths and A-cycles. An A-cycle is a cycle that contains exactly one vertex in A. Equivalently, an A-cycle may be considered to be an A-path whose endpoints are the same.

We note that if A and B are disjoint subsets of vertices, an A–B path is a path with one endpoint in A and the other in B and all internal vertices are not in $A \cup B$. All notation and terminology used is standard or defined as and when needed.

2 Three terminals

Theorem 1 *In every simple graph G with minimum degree $d \geq 2$, there are edges uv and vw such that G contains $\lfloor 3d/2 \rfloor$ edge-disjoint $\{u, v, w\}$-paths. If d is even, the paths can be chosen such that there are $d/2$ paths between each pair of vertices in $\{u, v, w\}$. If d is odd, the paths can be chosen such that there are $(d + 1)/2$ paths between any specified pair of vertices in $\{u, v, w\}$ and $(d - 1)/2$ paths between the other two pairs.*

The proof of Theorem 1 is based on the technique used by Mader in [2]. We introduce a few definitions in order to describe this technique.

Definition 2 *A sequence of distinct vertices (u_1, u_2, u_3) is said to be (a, b, c) edge-connected in a graph G if there are $a + b + c$ edge-disjoint $\{u_1, u_2, u_3\}$-paths in G, a of which have endpoints $\{u_1, u_2\}$, b have endpoints $\{u_1, u_3\}$ and c have endpoints $\{u_2, u_3\}$.*

An ordered clique K in a graph G is a complete subgraph of G with an ordering imposed on the vertices of K. We will be considering ordered pairs of the form (G, K), where K is an ordered clique in a graph G.

Definition 3 *Let (G, K) be any pair with K an ordered clique in a graph G. Let $(v_1, v_2, \ldots, v_k)$ be the ordering of vertices in K. If K is a proper subgraph of G, the* reduction *$\alpha(G, K)$ of the pair (G, K) is the pair (G', K') defined as follows:*

1. *If there is a vertex $v \in V(G) \setminus V(K)$ that is adjacent to all vertices in $V(K)$, then $G' = G$ and $V(K') = V(K) \cup \{v\}$ with the ordering $(v_1, v_2, \ldots, v_k, v)$ of vertices in K'. If there is more than one such vertex, choose any one arbitrarily.*

2. *Suppose no vertex in $V(G) \setminus V(K)$ is adjacent to all vertices in $V(K)$. For every vertex $u \in V(G) \setminus V(K)$, let $\pi(u)$ be the smallest index i such that u is not adjacent to $v_i \in V(K)$. Then $G' = (G - v_1) \cup \{uv_{\pi(u)} \mid u \in V(G) \setminus V(K), \pi(u) > 1\}$, and $K' = K - v_1$.*

The reduction step can be applied repeatedly to a pair (G, K), until $G - K$ is empty. Define $\alpha^0(G, K) = (G, K)$, and $\alpha^i(G, K) = \alpha(\alpha^{i-1}(G, K))$ for $i \geq 1$.

Some obvious properties of this reduction are noted in Lemma 4.

Lemma 4 *Let (G, K) be any pair and let $(G', K') = \alpha^i(G, K)$, for some $i \geq 0$. Then the following statements are true.*

1. *$G' - K'$ is an induced subgraph of $G - K$.*
2. *If every vertex in $G - K$ has degree at least d in G then every vertex in $G' - K'$ has degree at least d in G'.*
3. *For any set S of edges in $G' - K'$, $(G' - S, K') = \alpha^i(G - S, K)$.* □

Definition 5 *Let (G, K) be any pair with K a proper subgraph of G. A sequence of distinct vertices $(u_1, u_2, \ldots, u_m)$ in $G - K$ is said to be $(c_1, c_2, \ldots, c_m)$-joined to K in G, if there are $\sum_{i=1}^{m} c_i$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K)$ paths in G such that exactly c_i paths have u_i as an endpoint and no two paths have the same pair of endpoints.*

Lemma 6 *Let G be any graph and K an ordered clique in G that is a proper subgraph of G. Let $(G_i, K_i) = \alpha^i(G, K)$ and $(G_{i-1}, K_{i-1}) = \alpha^{i-1}(G, K)$, for some $i \geq 1$. Suppose a sequence of distinct vertices $(u_1, u_2, \ldots, u_m)$ in $G_i - K_i$ is $(c_1, c_2, \ldots, c_m)$-joined to K_i in G_i. If (G_i, K_i) is obtained by applying step 2 of the reduction to (G_{i-1}, K_{i-1}), then $(u_1, u_2, \ldots, u_m)$ is $(c_1, c_2, \ldots, c_m)$-joined to K_{i-1} in G_{i-1}.*

Proof: Let $\mathcal{P}_0$ be the set of $\sum_{j=1}^{m} c_j$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K_i)$ paths in G_i such that exactly c_j paths have u_j as an endpoint and no two paths have the same pair of endpoints. Since (G_i, K_i) is obtained by applying step 2 of the reduction to (G_{i-1}, K_{i-1}), $V(K_{i-1}) = V(K_i) \cup \{v_1\}$, $V(G_{i-1}) = V(G_i) \cup \{v_1\}$ and for some subset $X \subseteq V(G_i) \setminus V(K_i)$, $E(G_{i-1}) = (E(G_i) \cup \{uv_1 | u \in X\}) \setminus \{uv_{\pi(u)} | u \in X\})$. We call the edges in $B_0 = \{uv_{\pi(u)} | u \in X\}$ *bad* edges. If none of the paths in $\mathcal{P}_0$ contain a bad edge, then $\mathcal{P}_0$ is the required set of paths in G_{i-1}. Let l_0 be the largest index such that some edge in B_0 is incident with $v_{l_0} \in V(K_i)$. To prove the Lemma, we show that the paths in $\mathcal{P}_0$ can be modified so that none of the paths contain a bad edge. This modification is done in a sequence of steps. At each step we maintain a triple $(\mathcal{P}, B, l)$ satisfying the following properties:

1. B is a set of edges, called bad edges, joining vertices in $V(G_i) \setminus V(K_i)$ to vertices in $V(K_i)$. Any vertex $u \in V(G_i) \setminus V(K_i)$ is incident with at most one edge in B. If uv_p, for some $u \in V(G_i) \setminus V(K_i)$ and $v_p \in V(K_i)$, is an edge in B then uv_q is an edge in G_{i-1} for all $1 \leq q < p$. The largest index $j \geq 2$ such that some edge in B is incident with $v_j \in V(K_i)$ is denoted by l. Note that an edge in B may be parallel to an edge in G_{i-1} and we consider $G_{i-1} \cup B$ to be a multigraph.
2. $\mathcal{P}$ is a set of $\sum_{j=1}^{m} c_j$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K_i)$ paths in $G_{i-1} \cup B$ such that exactly c_j paths have u_j as an endpoint. If two paths in $\mathcal{P}$ have the same pair of endpoints, then one of the common endpoints must be v_l, and one of the two paths terminates at v_l with an edge in B while the other terminates with an edge not in B.

Note that $(\mathcal{P}_0, B_0, l_0)$ satisfies the two properties. Suppose $(\mathcal{P}, B, l)$ is a triple satisfying these properties with $l = 2$. If any path in $\mathcal{P}$ contains an edge $uv_2 \in B$, for some $u \in V(G_i) \setminus V(K_i)$, replace that edge by the edge uv_1 in G_{i-1}. This gives a new set of paths that are contained in G_{i-1}. Two paths terminating at v_1 cannot have their other endpoint common, as two paths in $\mathcal{P}$, having both endpoints common, cannot both terminate with a bad edge at v_2. This gives the required set of paths in G_{i-1}.

Suppose $l > 2$. We show that we can find a new triple with a smaller value of l. We may assume that every edge in B that is incident with v_l is contained in some path in $\mathcal{P}$, otherwise just delete the edge from B. Any two paths in $\mathcal{P}$ that terminate in v_l with a bad edge must have their other endpoints distinct. Let S be the set of indices j such that some path in $\mathcal{P}$ has $\{u_j, v_l\}$ as endpoints and contains a bad edge. We colour the bad edge in this path j. Thus all bad edges incident with v_l get distinct colours.

Define a subset $S' \subseteq S$ as the smallest subset of S satisfying the following:

1. If for some $j \in S$, there exists a path in $\mathcal{P}$ with endpoints $\{u_j, v_{l-1}\}$ that terminates in a bad edge incident with v_{l-1} then $j \in S'$.
2. Suppose $t \in S'$ and uv_l is the bad edge coloured t incident with v_l. If the edge uv_{l-1}, which is an edge in G_{i-1}, is contained in some path in $\mathcal{P}$ having u_j as an endpoint, and $j \in S$, then $j \in S'$.

Now for every bad edge uv_l incident with v_l, we do the following. If the edge uv_l is coloured j and $j \in S'$ then replace the edge uv_l by the good edge uv_{l-1} in the path in $\mathcal{P}$ that contains uv_l. If the edge uv_{l-1} was contained in some path in $\mathcal{P}$, add a new bad edge parallel to it, and replace uv_{l-1} by the bad edge that is parallel to it, in this path. If $j \notin S'$, add a new bad edge uv_{l-1} parallel to the good edge uv_{l-1}, and replace the edge uv_l by the new bad edge uv_{l-1} in the path in $\mathcal{P}$ that contains uv_l. Finally, delete the edge uv_l from B and add any newly added bad edge to B.

We claim that the new set of paths $\mathcal{P}'$ and the new set of bad edges B' satisfy the required properties. This gives a triple with a smaller value of l.

Since any new bad edge is obtained by replacing a bad edge of the form uv_l by uv_{l-1}, it is clear that the new set of bad edges satisfies the required property.

It remains to show that $\mathcal{P}'$ also satisfies the required property. From the construction, it can be seen that paths in $\mathcal{P}$ that terminate at v_l with a bad edge are replaced in $\mathcal{P}'$ by paths terminating at v_{l-1}, while all other paths have the same pair of endpoints. Hence v_{l-1} is the only vertex at which more than one path from some vertex u_j can terminate, and there can be at most two such paths.

Suppose there are two paths is $\mathcal{P}'$ with endpoints $\{u_j, v_{l-1}\}$ for some $j \in \{1, 2 \ldots, m\}$. Then one of the paths, say P_1, is obtained by replacing a bad edge uv_l of colour j, contained in some path in $\mathcal{P}$, either by the good edge uv_{l-1} or by a newly added bad edge uv_{l-1}. Hence $j \in S$. If $j \in S'$ then P_1 terminates with a good edge. By the definition of S', either the path in $\mathcal{P}$ with endpoints $\{u_j, v_{l-1}\}$ terminates with a bad edge, or if it terminates with a good edge, the good edge is replaced by a bad edge parallel to it. In either case, the second path in $\mathcal{P}'$ with endpoints $\{u_j, v_{l-1}\}$ terminates with a bad edge. If $j \notin S'$, P_1 terminates with a bad edge, and the other path in $\mathcal{P}'$ with endpoints $\{u_j, v_{l-1}\}$ is a path in $\mathcal{P}$. It must terminate with a good edge, by the definition of S'. Hence $(\mathcal{P}', B', l-1)$ is a triple satisfying the required properties. □

Lemma 7 *Let G be any graph and K an ordered clique in G that is a proper subgraph of G. Let $(G_i, K_i) = \alpha^i(G, K)$ and $(G_{i-1}, K_{i-1}) = \alpha^{i-1}(G, K)$, for some $i \geq 1$. Suppose a sequence of distinct vertices $(u_1, u_2, \ldots, u_m)$ in $G_i - K_i$ is $(c_1, c_2, \ldots, c_m)$-joined to K_i in G_i, and (G_i, K_i) is obtained by applying step 1 of the reduction to (G_{i-1}, K_{i-1}). Then there exists a vertex $v \in V(K_i)$ and a subset $S \subseteq \{1, 2, \ldots, m\}$ such that the following statements are true.*

1. *$G_{i-1} = G_i$ and $K_{i-1} = K_i - v$.*

2. *v is adjacent to all vertices in K_{i-1}.*

3. *$c_j > 0$ for all $j \in S$.*

4. *There are $|S|$ edge-disjoint $\{v\}$–$\{u_1, u_2, \ldots, u_m\}$ paths $\{Q_j | j \in S\}$ in $G_{i-1} - K_{i-1}$ such that Q_j has endpoints $\{u_j, v\}$, for all $j \in S$.*

5. *The sequence of vertices $(u_1, u_2, \ldots, u_m, v)$ is $(d_1, d_2, \ldots, d_m, d)$-joined to K_{i-1} in $G_{i-1} - (\bigcup_{j \in S} E(Q_j))$, where $d_j = c_j$ if $j \notin S$, $d_j = c_j - 1$ if $j \in S$ and $d = \max(d_1, d_2, \ldots, d_m)$.*

Proof: Let $\mathcal{P}$ be the set of $\sum_{j=1}^{m} c_j$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K_i)$ paths such that exactly c_j paths have u_j as an endpoint, and no two paths have the same pair of endpoints. Since (G_i, K_i) is obtained from (G_{i-1}, K_{i-1}) using step 1 of the reduction, $G_i = G_{i-1}$ and $V(K_i) = V(K_{i-1}) \cup \{v\}$, for some vertex v in $G_{i-1} - K_{i-1}$ that is adjacent to all vertices in K_{i-1}. This proves the first two statements in Lemma 7. Let $S \subseteq \{1, 2, \ldots, m\}$ be the subset of indices j such that some path in $\mathcal{P}$ has endpoints $\{u_j, v\}$. There can be at most one such path for each j and we denote these paths $\{Q_j | j \in S\}$. Clearly these paths are $\{v\}$–$\{u_1, u_2, \ldots, u_m\}$ paths contained in $G_{i-1} - K_{i-1}$. This proves the third and fourth statements in Lemma 7. Finally, $\mathcal{P}' = \mathcal{P} \setminus \{Q_j | j \in S\}$ is a collection of $\sum_{j=1}^{m} d_j$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K_{i-1})$ paths in $G_{i-1} - (\bigcup_{j \in S} E(Q_j))$ such that exactly d_j paths have u_j as an endpoint and no two paths have both endpoints the same. This implies that $|K_{i-1}| \geq d = \max(d_1, d_2, \ldots, d_m)$. Therefore, $\mathcal{P}' \cup \{vv_j | v_j \in V(K_{i-1}), 1 \leq j \leq d\}$ is a collection of $\sum_{j=1}^{m} d_j + d$ edge-disjoint $\{u_1, u_2, \ldots, u_m, v\}$–$V(K_{i-1})$ paths such that exactly d_j paths have u_j as an endpoint and no two paths have both endpoints the same. This completes the proof of Lemma 7. □

Lemma 8 *Let G be any graph, let K be an ordered clique in G and let $(u_1, u_2, \ldots, u_m, v)$ be a sequence of distinct vertices in $G - K$. Suppose for some subset $S' \subseteq \{1, 2, \ldots, m\}$, $G - K$ contains $|S'|$ edge-disjoint $\{v\}$–$\{u_1, u_2, \ldots, u_m\}$ paths $\{Q_i | i \in S'\}$, with Q_i having endpoints $\{u_i, v\}$ for all $i \in S'$, such that $(u_1, u_2, \ldots, u_m, v)$ is $(c_1, c_2, \ldots, c_m, c)$-joined to K in $G - (\bigcup_{i \in S'} E(Q_i))$. Let $S'_k = \{i | i \in S', c_i \geq c - k\}$. If $|S'_k| \leq k$ for all $0 \leq k \leq |S'|$, then $(u_1, u_2, \ldots, u_m)$ is $(d_1, d_2, \ldots, d_m)$-joined to K in G, where $d_i = c_i$ if $i \notin S'$ and $d_i = c_i + 1$ if $i \in S'$.*

Proof: Let $\mathcal{P}$ be the set of $\sum_{i=1}^{m} c_i + c$ edge-disjoint $\{u_1, u_2, \ldots, u_m, v\}$–$V(K)$ paths in $G - (\bigcup_{i \in S'} E(Q_i))$ such that exactly c_i paths have endpoint u_i and no two paths have the same pair of endpoints. Let V_i be the subset of vertices in $V(K)$ that are endpoints of paths is $\mathcal{P}$ having u_i as one endpoint, for $1 \le i \le m$. Similarly, let V be the set of endpoints in $V(K)$ of paths in $\mathcal{P}$ that have v as an endpoint. Note that $|V_i| = c_i$ and $|V| = c$.

Let the vertices $\{u_i | i \in S'\}$ be ordered $(u_{i_1}, u_{i_2}, \ldots, u_{i_r})$ such that $c_{i_1} \ge c_{i_2} \ge \cdots \ge c_{i_r}$. Since $|S'_k| \le k$ for all $0 \le k \le r$, $c_{i_k} \le c - k$ for $1 \le k \le r$. Hence, there exist distinct vertices $(v_{i_1}, v_{i_2}, \ldots, v_{i_r})$ such that $v_{i_k} \in V$ and $v_{i_k} \notin V_{i_k}$, for $1 \le k \le r$. Let Q'_{i_k} be the path in $\mathcal{P}$ with endpoints $\{v, v_{i_k}\}$, and let P_{i_k} be a path with endpoints $\{u_{i_k}, v_{i_k}\}$ contained in $Q_{i_k} \cup Q'_{i_k}$ for $1 \le k \le r$. Then $(\mathcal{P} \cup \{P_{i_k} | 1 \le k \le r\}) \setminus \{Q'_{i_k} | 1 \le k \le r\}$ contains a collection of $\sum_{i=1}^{m} d_i$ edge-disjoint $\{u_1, u_2, \ldots, u_m\}$–$V(K)$ paths such that exactly d_i paths have endpoint u_i and no two of the paths have the same pair of endpoints, where $d_i = c_i$ if $i \notin S'$ and $d_i = c_i + 1$ if $i \in S'$. □

If Lemma 8 holds for some subset S' and paths $\{Q_i | i \in S'\}$, we say the paths $\{Q_i | i \in S'\}$ can be *extended* to the clique K in G.

Lemma 9 *Let $(G_i, K_i) = \alpha^i(G, \emptyset)$ for some $i \ge 0$ and let (u_1, u_2, u_3) be a sequence of distinct vertices in $G_i - K_i$. Then the following statements are true for all integers $k \ge 1$.*

1. *If (u_1, u_2, u_3) is $(2k, 2k, 2k)$-joined to K_i in G_i, then (u_1, u_2, u_3) is (k, k, k) edge-connected in G.*

2. *If (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-2)$-joined to K_i in G_i, then (u_1, u_2, u_3) is $(k, k-1, k-1)$ edge-connected in G.*

3. *If (u_1, u_2, u_3) is $(2k, 2k, 2k-1)$-joined to K_i in G_i, then (u_1, u_2, u_3) is $(k+1, k-1, k-1)$ edge-connected in G, and also $(k, k, k-1)$ edge-connected in G.*

4. *If $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_2, P_3 with endpoints $\{u_2, u_4\}$ and $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k, 2k-1, 2k-1, 2k)$-joined to K_i in $G_i - (E(P_2) \cup E(P_3))$, then (u_1, u_2, u_3) is (k, k, k) edge-connected in G.*

5. *If $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_1, P_2, P_3 with endpoints $\{u_1, u_4\}, \{u_2, u_4\}$ and $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_i in $G_i - (E(P_1) \cup E(P_2) \cup E(P_3))$, then (u_1, u_2, u_3) is (k, k, k) edge-connected in G.*

6. *If $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_2, P'_2, P_3 with endpoints $\{u_2, u_4\}$, $\{u_2, u_4\}$ and $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k, 2k-2, 2k-1, 2k)$-joined to K_i in $G_i - (E(P_2) \cup E(P'_2) \cup E(P_3))$, then (u_1, u_2, u_3) is (k, k, k) edge-connected in G.*

Proof: The proof is by induction on i, the number of steps in the reduction. If $i = 0$, then $G_i = G$ and K_i is empty, and the Lemma is trivially true, since no vertex in $G_i - K_i$ can be joined by a path to K_i, and there do not exist vertices (u_1, u_2, u_3) satisfying the hypothesis of any of the statements in Lemma 9.

Suppose $i > 0$ and let $(G_{i-1}, K_{i-1}) = \alpha^{i-1}(G, \emptyset)$. Then (G_i, K_i) is obtained by applying either step 1 or step 2 of the reduction, defined in Definition 3, to (G_{i-1}, K_{i-1}).

Suppose (G_i, K_i) is obtained from (G_{i-1}, K_{i-1}) by applying step 2 of the reduction. By Lemma 6, if a sequence of vertices (u_1, u_2, u_3) in $G_i - K_i$ satisfies the hypothesis of any of the statements in Lemma 9 in G_i, then (u_1, u_2, u_3) satisfies the same hypothesis in G_{i-1}. Hence, by induction, (u_1, u_2, u_3) satisfies the corresponding conclusion in G, and each of the statements in the Lemma is true.

Suppose (G_i, K_i) is obtained from (G_{i-1}, K_{i-1}) by applying step 1 of the reduction. Then $G_{i-1} = G_i$ and $K_{i-1} = K_i - v$ for some vertex v that satisfies the statements in Lemma 7. Let $S \subseteq \{1, 2, 3, 4\}$ be the subset that satisfies the statements in Lemma 7 and let $\{Q_j | j \in S\}$ be the corresponding paths. If S is empty, by statement 5 in Lemma 7, if (u_1, u_2, u_3) satisfies the hypothesis of any statement in Lemma 9 in G_i, it satisfies the same hypothesis in G_{i-1}, and we can apply induction. We may therefore assume S is not empty.

We consider each statement in Lemma 9 separately.

Case 1. Suppose (u_1, u_2, u_3) is $(2k, 2k, 2k)$-joined to K_i in G_i.

If $|S| = 1$, by Lemmas 7 and 8, with $S' = S$, we can extend the path $\{Q_j | j \in S\}$ to the clique K_{i-1} in G_{i-1}. Hence, (u_1, u_2, u_3) is $(2k, 2k, 2k)$-joined to K_{i-1} in G_{i-1}, and by induction, using statement 1 in Lemma 9, it is (k, k, k) edge-connected in G.

Suppose $|S| = 2$ and without loss of generality, $S = \{2, 3\}$. Considering v to be the vertex u_4 and Q_2, Q_3 to be the paths P_2, P_3, (u_1, u_2, u_3, u_4) is $(2k, 2k-1, 2k-1, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_3))$, by Lemma 7. By induction, using statement 4 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G. It is worth noting that in this case, (u_1, u_2, u_3) is $(2k, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q_2) \cup E(Q_3))$. Unfortunately, this does not imply it is $(k, k, k-1)$ edge-connected in $G - (E(Q_2) \cup E(Q_3))$.

If $S = \{1, 2, 3\}$ then considering v to be the vertex u_4, and the paths Q_1, Q_2, Q_3 to be the paths P_1, P_2, P_3 respectively, (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$, by Lemma 7. By induction, using statement 5 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

Case 2. Suppose (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-2)$-joined to K_i in G_i.

If $|S| \leq 2$ and $S \neq \{1, 2\}$, then by Lemma 8 with $S' = S$, we can extend the paths $\{Q_j | j \in S\}$ to K_{i-1} in G_{i-1}, and hence (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-2)$-joined to K_{i-1} in G_{i-1}. By induction, using statement 2 in Lemma 9, (u_1, u_2, u_3) is $(k, k-1, k-1)$ edge-connected in G.

If $S = \{1, 2\}$, let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_2$. Then, by Lemma 7, (u_1, u_2, u_3) is $(2k-2, 2k-2, 2k-2)$-joined to K_{i-1} in $G_{i-1} - E(Q)$. If $k = 1$ then Q is the required path in G with endpoints $\{u_1, u_2\}$. If $k > 1$, by induction, using statement 1 in Lemma 9, (u_1, u_2, u_3) is $(k-1, k-1, k-1)$ edge-connected in $G - E(Q)$, and hence $(k, k-1, k-1)$ edge-connected in G. If $S = \{1, 2, 3\}$, the same argument holds, as by Lemma 8 with $S' = \{3\}$, we can extend the path Q_3 to K_{i-1} in $G_{i-1} - E(Q)$, and hence (u_1, u_2, u_3) is $(2k-2, 2k-2, 2k-2)$-joined to K_{i-1} in $G_{i-1} - E(Q)$.

Case 3. Suppose (u_1, u_2, u_3) is $(2k, 2k, 2k-1)$-joined to K_i in G_i.

If $\{1, 2\} \not\subseteq S$ then by Lemma 8 with $S' = S$, we can extend the paths $\{Q_j | j \in S\}$ to the clique K_{i-1} in G_{i-1}, and hence (u_1, u_2, u_3) is $(2k, 2k, 2k-1)$-joined to K_{i-1} in G_{i-1}. By induction, using statement 3 in Lemma 9, (u_1, u_2, u_3) is $(k+1, k-1, k-1)$ edge-connected in G, and also $(k, k, k-1)$ edge-connected in G.

If $S = \{1,2\}$, let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_2$. Then (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - E(Q)$. By induction, using statement 2 in Lemma 9, (u_1, u_2, u_3) is $(k, k-1, k-1)$ edge-connected in $G - E(Q)$, and also $(k-1, k, k-1)$ edge-connected in $G - E(Q)$. Hence (u_1, u_2, u_3) is $(k+1, k-1, k-1)$ edge-connected in G, and also $(k, k, k-1)$ edge-connected in G. The same argument can be used if $S = \{1,2,3\}$, as by applying Lemma 8 with $S' = \{3\}$, we can extend the path Q_3 to the clique K_{i-1} in $G_{i-1} - E(Q)$, and hence (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - E(Q)$.

Case 4. Suppose $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_2, P_3 with endpoints $\{u_2, u_4\}$, $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k, 2k-1, 2k-1, 2k)$-joined to K_i in $G_i - (E(P_2) \cup E(P_3))$.

If $|S| \leq 2$ and $S \neq \{1,4\}$ then by Lemma 8 with $S' = S$, we can extend the paths $\{Q_j | j \in S\}$ to the clique K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_3))$, and hence (u_1, u_2, u_3, u_4) is $(2k, 2k-1, 2k-1, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_3))$. By induction, using statement 4 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

If $S = \{1,4\}$, let P_1 be a path with endpoints $\{u_1, u_4\}$ contained in $Q_1 \cup Q_4$. Then (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$. By induction, using statement 5 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G. The same argument holds if $S = \{1,2,4\}$ or $S = \{1,3,4\}$, as by Lemma 8 with $S' = \{2\}$ or $S' = \{3\}$ respectively, we can extend the path Q_2 or Q_3 to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$, and hence (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$.

If $S = \{1,2,3\}$ then by Lemma 7, (u_1, u_2, u_3, u_4, v) is $(2k-1, 2k-2, 2k-2, 2k, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_3) \cup E(Q_1) \cup E(Q_2) \cup E(Q_3))$. Applying Lemma 8 with $S' = \{2,3\}$, we can extend the paths $\{P_2, P_3\}$ to K_{i-1} in $G_{i-1} - (E(Q_1) \cup E(Q_2) \cup E(Q_3))$. Hence Q_1, Q_2, Q_3 are edge-disjoint $\{v\}$–$\{u_1, u_2, u_3\}$ paths in $G_{i-1} - K_{i-1}$ with endpoints $\{u_1, v\}$, $\{u_2, v\}$ and $\{u_3, v\}$, such that (u_1, u_2, u_3, v) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q_1) \cup E(Q_2) \cup E(Q_3))$. By induction, considering v to be the vertex u_4 in statement 5 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

If $S = \{2,3,4\}$ then (u_1, u_2, u_3, u_4, v) is $(2k, 2k-2, 2k-2, 2k-1, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_3) \cup E(Q_2) \cup E(Q_3) \cup E(Q_4))$. Let P_2' be a path with endpoints $\{u_2, v\}$ contained in $P_2 \cup Q_4$. Applying Lemma 8 with $S' = \{3\}$, we can extend the path P_3 to K_{i-1} in $G_{i-1} - (E(P_2') \cup E(Q_2) \cup E(Q_3))$, and hence P_2', Q_2, Q_3 are edge-disjoint $\{v\}$–$\{u_1, u_2, u_3\}$ paths in $G_{i-1} - K_{i-1}$ with endpoints $\{u_2, v\}$, $\{u_2, v\}$ and $\{u_3, v\}$ respectively, such that (u_1, u_2, u_3, v) is $(2k, 2k-2, 2k-1, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2') \cup E(Q_2) \cup E(Q_3))$. By induction, considering v to be the vertex u_4 in statement 6 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

Finally, suppose $S = \{1,2,3,4\}$. Let Q be a path with endpoints $\{u_1, u_3\}$ contained in $Q_1 \cup Q_3$ and Q' a path with endpoints $\{u_2, u_3\}$ contained in $P_2 \cup P_3$. Applying Lemma 8 with $S' = \{2\}$, we can extend the path Q_2 to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$, hence (u_1, u_2, u_3) is $(2k-1, 2k-1, 2k-2)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. By induction, using statement 2 in Lemma 9, (u_1, u_2, u_3) is $(k, k-1, k-1)$ edge-connected in $G - (E(Q) \cup E(Q'))$ and hence (k, k, k) edge-connected in G.

Case 5. Suppose $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_1, P_2, P_3 with endpoints $\{u_1, u_4\}, \{u_2, u_4\}$ and $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_i in $G_i - (E(P_1) \cup E(P_2) \cup E(P_3))$.

If $|S| = 1$, by Lemma 8 with $S' = S$, we can extend the path $\{Q_j, j \in S\}$ to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$, hence (u_1, u_2, u_3, u_4) is $(2k-1, 2k-1, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(P_1) \cup E(P_2) \cup E(P_3))$. By induction, using statement 5 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

Suppose $S = \{1, 2\}$. Let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_2$ and let Q' be a path with endpoints $\{u_2, u_3\}$ contained in $P_2 \cup P_3$. Applying Lemma 8 with $S' = \{1\}$, we can extend the path P_1 to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. Hence, (u_1, u_2, u_3) is $(2k-1, 2k-2, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. By induction, using statement 2 in Lemma 9, $(u_1, u_2, u_3\}$ is $(k-1, k, k-1)$ edge-connected in $G - (E(Q) \cup E(Q'))$ and hence (k, k, k) edge-connected in G. A similar argument holds if $S = \{1, 3\}$ or $S = \{2, 3\}$, by symmetry. If $S = \{1, 2, 3\}$, apply Lemma 8 twice, with $S' = \{1\}$ and extend the path P_1 to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q') \cup E(Q_3))$, and again with $S' = \{3\}$, extend the path Q_3 to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. Hence (u_1, u_2, u_3) is $(2k-1, 2k-2, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$ and the previous argument holds.

Suppose $S = \{1, 4\}$. Let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_4 \cup P_2$ and Q' a path with endpoints $\{u_1, u_3\}$ contained in $P_1 \cup P_3$. Then (u_1, u_2, u_3) is $(2k-2, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$ and by induction, using statement 2 in Lemma 9, (u_1, u_2, u_3) is $(k-1, k-1, k)$ edge-connected in $G - (E(Q) \cup E(Q'))$. Hence (u_1, u_2, u_3) is (k, k, k) edge-connected in G. A similar argument holds if $S = \{2, 4\}$ or $S = \{3, 4\}$, by symmetry. Further, if $S = \{1, 2, 4\}$, applying Lemma 8 with $S' = \{2\}$, we can extend the path Q_2 to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$, hence (u_1, u_2, u_3) is $(2k-2, 2k-1, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. The same argument holds if $S = \{1, 3, 4\}$ or $S = \{2, 3, 4\}$, by symmetry.

Finally, suppose $S = \{1, 2, 3, 4\}$. Let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_4 \cup P_2$, let Q' be a path with endpoints $\{u_1, u_3\}$ contained in $P_1 \cup P_3$, and Q'' a path with endpoints $\{u_2, u_3\}$ contained in $Q_2 \cup Q_3$. Then (u_1, u_2, u_3) is $(2k-2, 2k-2, 2k-2)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q') \cup E(Q''))$. If $k = 1$, Q, Q' and Q'' are the required paths in G. If $k > 1$, by induction, using statement 1 in Lemma 9, (u_1, u_2, u_3) is $(k-1, k-1, k-1)$ edge-connected in $G - (E(Q) \cup E(Q') \cup E(Q''))$ and hence is (k, k, k) edge-connected in G.

Case 6. Suppose $G_i - K_i$ contains a vertex $u_4 \notin \{u_1, u_2, u_3\}$ and edge-disjoint $\{u_4\}$–$\{u_1, u_2, u_3\}$ paths P_2, P_2', P_3 with endpoints $\{u_2, u_4\}, \{u_2, u_4\}$ and $\{u_3, u_4\}$ respectively, such that (u_1, u_2, u_3, u_4) is $(2k, 2k-2, 2k-1, 2k)$-joined to K_i in $G_i - (E(P_2) \cup E(P_2') \cup E(P_3))$.

If $\{1, 4\} \not\subseteq S$, applying Lemma 8 with $S' = S$ and extending the paths $\{Q_j | j \in S\}$ to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_2') \cup E(P_3))$, we see that (u_1, u_2, u_3, u_4) is $(2k, 2k-2, 2k-1, 2k)$-joined to K_{i-1} in $G_{i-1} - (E(P_2) \cup E(P_2') \cup E(P_3))$. By induction, using statement 6 in Lemma 9, (u_1, u_2, u_3) is (k, k, k) edge-connected in G.

Suppose $S = \{1, 4\}$. Let Q be a path with endpoints $\{u_1, u_2\}$ contained in $Q_1 \cup Q_4 \cup P_2$ and let Q' be a path with endpoints $\{u_2, u_3\}$ contained in $P_2' \cup P_3$. Then (u_1, u_2, u_3) is $(2k-1, 2k-2, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. By induction, using statement 2 in Lemma 9, (u_1, u_2, u_3) is $(k-1, k, k-1)$ edge-connected in $G - (E(Q) \cup E(Q'))$, and hence is (k, k, k) edge-connected in G. The same argument holds for any set S such that $\{1, 4\} \subseteq S$, as we can apply Lemma 8 with $S' = S \cap \{2, 3\}$ and extend the paths $\{Q_j | j \in S'\}$ to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$, and get that (u_1, u_2, u_3) is $(2k-1, 2k-2, 2k-1)$-joined to K_{i-1} in $G_{i-1} - (E(Q) \cup E(Q'))$. □

Lemma 10 *Let G be a graph with minimum degree $d \geq 2$. Then there exists a pair $(G_i, K_i) = \alpha^i(G, \emptyset)$ such that one of the following is true.*

1. *$G_i - K_i$ contains two edges $\{uv, vw\}$ such that (u, v, w) is $(d-1, d-2, d-1)$-joined to K_i in $G_i - \{uv, vw\}$.*

2. *$G_i - K_i$ contains three edges $\{uv, uw, vw\}$ such that (u, v, w) is $(d-2, d-2, d-2)$-joined to K_i in $G_i - \{uv, uw, vw\}$.*

Proof: Let $i+1$ be the smallest integer such that $G_{i+1} - K_{i+1}$ has maximum degree 1, that is, each component of $G_{i+1} - K_{i+1}$ is either K_1 or K_2. Since $G_0 - K_0$ is G, which has minimum degree at least 2, and the reduction terminates when $G_i - K_i$ is empty, there exists such an $i \geq 0$. We must have $G_i = G_{i+1}$ and $K_i = K_{i+1} - v$ for some vertex v, by the minimality of i. Every vertex in $G_{i+1} - K_{i+1}$ has degree at most two in $G_i - K_i$ and hence $|K_i| \geq d-2$. Further, v is adjacent to every vertex in K_i, by Lemma 7.

Suppose v is adjacent to both endpoints of an edge uw in $G_{i+1} - K_{i+1}$. Then $\{uv, uw, vw\}$ are edges in $G_i - K_i$ such that (u, v, w) is $(d-2, d-2, d-2)$-joined to K_i in $G_i - \{uv, uw, vw\}$.

Suppose v is adjacent to a vertex u that is an endpoint of an edge uw in $G_{i+1} - K_{i+1}$, but is not adjacent to w. Then w has at least $d-1$ neighbours in K_i and hence so does v. Thus $\{uv, uw\}$ are edges in $G_i - K_i$ such that (v, u, w) is $(d-1, d-2, d-1)$-joined to K_i in $G_i - \{uv, uw\}$.

The only other possibility is that v is the only vertex of degree at least two in $G_i - K_i$, and is adjacent to vertices u, w that are isolated in $G_{i+1} - K_{i+1}$. Then $\{uv, vw\}$ are edges in $G_i - K_i$ such that (u, v, w) is $(d-1, d-2, d-1)$-joined to K_i in $G_i - \{uv, vw\}$. □

Proof: (Theorem 1) Suppose $d = 2k$ is even. By Lemma 10, there exists an $i \geq 0$ such that $(G_i, K_i) = \alpha^i(G, \emptyset)$ satisfies one of the statements in Lemma 10.

Suppose there exist edges $\{uv, vw\}$ in $G_i - K_i$ such that (u, v, w) is $(2k-1, 2k-2, 2k-1)$-joined to K_i in $G_i - \{uv, vw\}$. By Lemma 9, statement 2, (u, v, w) is $(k-1, k, k-1)$ edge-connected in $G - \{uv, vw\}$ and hence (k, k, k) edge-connected in G.

Suppose there exist edges $\{uv, vw, uw\}$ in $G_i - K_i$ such that (u, v, w) is $(2k-2, 2k-2, 2k-2)$-joined to K_i in $G_i - \{uv, vw, uw\}$. If $k = 1$, the three edges form the required paths. If $k > 1$, by Lemma 9, statement 1, (u, v, w) is $(k-1, k-1, k-1)$ edge-connected in $G - \{uv, vw, uw\}$ and hence (k, k, k) edge-connected in G.

A similar argument holds if $d = 2k+1$ is odd. Suppose there exist edges $\{uv, vw\}$ in $G_i - K_i$ such that (u, v, w) is $(2k, 2k-1, 2k)$-joined to K_i in $G_i - \{uv, vw\}$. By Lemma 9, statement 3, (u, v, w) is $(k-1, k+1, k-1)$ edge-connected in $G - \{uv, vw\}$, as well as $(k, k, k-1)$ edge-connected in $G - \{uv, vw\}$. Hence (u, v, w) is $(k, k+1, k)$ edge-connected in G, as well as $(k+1, k, k)$ edge-connected in G.

Suppose there exist edges $\{uv, vw, uw\}$ in $G_i - K_i$ such that (u, v, w) is $(2k-1, 2k-1, 2k-1)$-joined to K_i in $G_i - \{uv, vw, uw\}$. By Lemma 9, statement 2, (u, v, w) is $(k-1, k, k-1)$ and also $(k, k-1, k-1)$ edge-connected in $G - \{uv, vw, uw\}$. Hence (u, v, w) is $(k, k+1, k)$ as well as $(k+1, k, k)$ edge-connected in G. □

3 Multigraphs

Theorem 11 *Let k, d be positive integers and G a multigraph of order at least k and minimum degree at least d. Then there exists a set A of k vertices such that G contains $\lfloor dk/2 \rfloor$ edge-disjoint $[A]$-paths, where an $[A]$-path is either an A-path or an A-cycle.*

Proof: It is sufficient to consider the case when G is connected. If not, let $C_1, C_2, \ldots, C_m$ be the connected components of G. Let i be the smallest integer such that $|C_1| + |C_2| + \cdots + |C_i| \geq k$. If $i = 1$, we consider only the component C_1. If $i > 1$, choose A to be $V(C_1) \cup \cdots \cup V(C_{i-1}) \cup A'$, where A' is a set of $k - (|C_1| + \cdots + |C_{i-1}|)$ vertices in C_i such that there are $\lfloor d|A'|/2 \rfloor$ edge-disjoint $[A']$-paths in C_i. Then the total number of $[A]$-paths in G is

$$\begin{aligned} &\geq \lceil d|C_1|/2 \rceil + \cdots + \lceil d|C_{i-1}|/2 \rceil + \lfloor d(k - (|C_1| + \cdots + |C_{i-1}|))/2 \rfloor \\ &\geq \lfloor dk/2 \rfloor. \end{aligned}$$

Suppose G is connected and has order $n \geq k$. If $n = k$ then the edges in G are the required paths, so we may assume $n > k$. Order the vertices in G $(v_1, v_2, \ldots, v_n)$ such that v_i is adjacent to at least one vertex v_j with $j > i$ for $1 \leq i < n$. Let A_i be the set of vertices $\{v_1, v_2, \ldots, v_i\}$ and $B_i = V(G) \setminus A_i$. We claim that there are $\lfloor dk/2 \rfloor$ edge-disjoint $[A_k]$-paths in G.

To prove this, we show that for each i, $k \leq i < n$, G contains a set of $[A_k]$-paths $\mathcal{P}_i$, and a set of A_k–B_i paths $\mathcal{Q}_i$, such that the paths in $\mathcal{P}_i \cup \mathcal{Q}_i$ are edge-disjoint, and $|\mathcal{Q}_i| \geq dk - 2|\mathcal{P}_i|$.

For $i = k$, let $\mathcal{P}_k$ be the set of edges with both endpoints in A_k and let $\mathcal{Q}_k$ be the set of edges that join a vertex in A_k to a vertex in B_k.

Suppose for some i, $k \leq i < n - 1$, we have the sets of paths $\mathcal{P}_i$ and $\mathcal{Q}_i$. Let $\mathcal{Q}$ be the subset of paths in $\mathcal{Q}_i$ that terminate in v_{i+1}. Let $\{P_1, P_2, \ldots, P_m\}$ be the paths in $\mathcal{Q}$. Let Q_j be a $[A_k]$-path that is contained in $P_{2j-1} \cup P_{2j}$, for $1 \leq j \leq \lfloor m/2 \rfloor$. Let $\mathcal{P}_{i+1} = \mathcal{P}_i \cup \{Q_j | 1 \leq j \leq \lfloor m/2 \rfloor\}$. If m is even, let $\mathcal{Q}_{i+1} = \mathcal{Q}_i \setminus \mathcal{Q}$. If m is odd, let Q be the A_k–B_{i+1} path contained in $P_m \cup \{v_{i+1}v_l\}$, where v_l is a vertex adjacent to v_{i+1} with $l > i + 1$. Now let $\mathcal{Q}_{i+1} = (\mathcal{Q}_i \setminus \mathcal{Q}) \cup \{Q\}$. Then $|\mathcal{P}_{i+1}| = |\mathcal{P}_i| + \lfloor m/2 \rfloor$ and $|\mathcal{Q}_{i+1}| = |\mathcal{Q}_i| - 2\lfloor m/2 \rfloor$. By induction, $|\mathcal{Q}_{i+1}| \geq dk - 2|\mathcal{P}_{i+1}|$.

Applying the same argument to paths in $\mathcal{Q}_{n-1}$, which must terminate in v_n, we get $|\mathcal{P}_{n-1}| + \lfloor |\mathcal{Q}_{n-1}|/2 \rfloor$ edge-disjoint $[A_k]$-paths in G. Since $|\mathcal{Q}_{n-1}| \geq dk - 2|\mathcal{P}_{n-1}|$, there are at least $\lfloor dk/2 \rfloor$ edge-disjoint $[A_k]$-paths in G. □

References

[1] W. Mader, Minimale n-fach kantenzusammenhängende Graphen, Math. Ann 191 (1971), 21-28.

[2] W. Mader, Existenz gewisser Konfigurationen in n-gesättigten Graphen und in Graphen genügend großer Kantendichte, Math. Ann. 194 (1971), 295-312.

[3] W. Mader, Über die Maximalzahl kantendisjunktur A-Wege, Arch. Math. (Basel), 30 (1978), 327-336.

[4] W. Mader, Über die Maximalzahl kreuzungsfreier H-Wege, Arch. Math. (Basel), 31 (1978), 387-402.

[5] A. Schrijver, A short proof of Mader's $\mathcal{S}$-paths theorem, J. Combin. Theory Ser. B 82 (2001), 319-321.