Explicit Δ-edge-coloring of adjacent levels in a divisor lattice

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Abstract

We show that the explicit 1-factorization of the middle levels of the Boolean lattice, defined by Duffus, Kierstead and Snevily [J. Comb. Theory Ser. A 65 (1994) 334–342] can be generalized in a simple way to define an explicit Δ-edge-coloring of any adjacent levels in any divisor lattice. The 1-factorization defined by Kierstead and Trotter [Order 5 (1988) 163–171] can also be generalized to define a different Δ-edge-coloring of the middle levels in any divisor lattice.

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1. Introduction

One of the well-known properties of a finite Boolean lattice is the existence of a symmetric chain decomposition. This was first proved by de Bruijn, Tengbergen, and Kruyswijk [1], using an inductive argument. They in fact showed that the property holds for the more general lattice of divisors of a number, ordered by divisibility, or equivalently the lattice of multisubsets of a multiset, ordered by inclusion. Greene and Kleitman [3] gave an explicit description of this decomposition, and showed that the same description applies to all divisor lattices as well.

A symmetric chain decomposition implies the existence of a matching between elements in two adjacent levels in the lattice, such that all elements in the level with fewer elements are matched to comparable elements in the

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other level. Kierstead and Trotter [6] defined a collection of pairwise disjoint perfect matchings in the middle levels of a Boolean lattice, called lexical matchings, which form a 1-factorization of the middle levels graph. Duffus, Kierstead and Snevily [2] gave an alternative explicit 1-factorization, called the modular 1-factorization, and showed that it is different from the lexical 1-factorization. Recently, in [5], different interpretations of these 1-factorizations were given, in an attempt to extend these to 1-factorizations of the bipartite Kneser graph, which is the comparability graph of elements in any two symmetric levels of the Boolean lattice. However, that appears to be a challenging problem and only some partial results have been obtained.

In this note, we extend these results to divisor lattices. Since the comparability graph of elements in two adjacent levels in a divisor lattice need not be regular, a 1-factorization may not exist. However, since it is bipartite, it is Δ-edge-colorable, where Δ is the maximum degree.

We show that the definition of the modular 1-factorization can be naturally extended to give an explicit Δ-edge-coloring of the comparability graph of elements in any two adjacent levels in any divisor lattice. Also, the definition of the lexical 1-factorization can be extended to give an explicit Δ-edge-coloring of the middle levels in any divisor lattice.

Instead of divisors of a number ordered by divisibility, we consider the equivalent lattice of multisubsets of a multiset, ordered by the subset relation. For any positive integer m, let $[m]$ denote the set of numbers $\{1, 2, \ldots, m\}$ with the operation of addition modulo m defined on it. For distinct elements $i, j \in [m]$, let $[i, j]$ denote the set of numbers $\{i, i+1, \ldots, j\}$, where addition is modulo m. Let $[i, j) = [i, j] \setminus \{j\}$, $(i, j] = [i, j] \setminus \{i\}$ and $(i, j) = [i, j] \setminus \{i, j\}$. A multiset M over $[m]$ is an m-tuple $(c_1, c_2, \ldots, c_m)$ of non-negative integers. The size or cardinality of M, denoted $|M|$, is $\sum_{i=1}^{m} c_i$. A multiset $A = (a_1, a_2, \ldots, a_m)$ is a subset of a multiset $B = (b_1, b_2, \ldots, b_m)$ if $a_i \leq b_i$ for all $1 \leq i \leq m$.

We consider the lattice of all multisubsets of a fixed multiset $M = (c_1, c_2, \ldots, c_m)$ over $[m]$ of size $n = \sum_{i=1}^{m} c_i$, ordered by the subset relation, for any integers $m \geq 1$ and $c_i > 0$, for $1 \leq i \leq m$. For simplicity, we will henceforth refer to multisets and multisubsets as just sets and subsets. If $A = (a_1, a_2, \ldots, a_m)$ is a subset of M, then $A^c = (c_1 - a_1, c_2 - a_2, \ldots, c_m - a_m)$ is the complement of A. An element $i \in [m]$ belongs to the subset A, denoted $i \in A$, if $a_i > 0$. If $i \in A$, let $A \setminus \{i\}$ be the subset $(a_1, a_2, \ldots, a_i - 1, \ldots, a_m)$. If $i \in A^c$, then $A \cup \{i\}$ denotes the subset $(a_1, a_2, \ldots, a_i + 1, \ldots, a_m)$.

The kth level of the lattice is the set of all subsets of M of size k, denoted

$\mathcal{M}_k$. The bipartite graph with vertex set $\mathcal{M}_k \cup \mathcal{M}_{k+1}$, with subset $A \in \mathcal{M}_k$ adjacent to a subset $B \in \mathcal{M}_{k+1}$ iff $A \subset B$, is called the kth level graph of the lattice. If n is odd, the $((n-1)/2)$th level graph is called the middle level. We only need to consider the case $k < n/2$, since the kth level graph is isomorphic to the $(n-1-k)$th level graph, by considering complements.

Lemma 1. *The maximum degree of the kth level graph is* $\min(m, n-k)$.

Proof. If A is a subset in $\mathcal{M}_k$, there are at most $\min(m, n-k)$ elements $i \in [m]$ such that $i \in A^c$, and A is adjacent to at most $\min(m, n-k)$ sets $A \cup \{i\}$ in $\mathcal{M}_{k+1}$. If B is any subset in $\mathcal{M}_{k+1}$, there are at most $\min(m, k+1)$ elements $i \in [m]$ such that $i \in B$. Since $k < n/2$, $n-k \geq k+1$, and B is adjacent to at most $\min(m, n-k)$ sets $B \setminus \{i\}$ in $\mathcal{M}_k$. Therefore the maximum degree is at most $\min(m, n-k)$.

To show that the bound is always achieved, consider the subset $M' = (c_1 - 1, c_2 - 1, \ldots, c_m - 1)$ of M of size $n-m$. If $m \leq n-k$, any subset $A \subseteq M'$ of size k is adjacent to $A \cup \{i\}$ for all $i \in [m]$, since $i \in A^c$. Therefore the degree of A is m. If $n-k \leq m$ then $|M'| \leq k$ and any subset $A \supseteq M'$ in $\mathcal{M}_k$ is adjacent to $n-k$ subsets in $\mathcal{M}_{k+1}$. Therefore the degree of A is $n-k$. $\square$

It is easy to construct a proper m-edge-coloring of any adjacent levels by assigning the color i to the edge from a set $A \in \mathcal{M}_k$ to $A \cup \{i\}$, for any $i \in A^c$. We only need to consider the case $n-k < m$ and show that there exists a proper $(n-k)$-edge-coloring of the kth level graph.

2. Modular matchings

In this section, we generalize the modular matchings in the middle levels of the Boolean lattice to define an explicit $(n-k)$-edge-coloring of the kth level graph in the multiset lattice.

Let $w(i) = \sum_{j=1}^{i} c_j$, for $1 \leq i \leq m$. Thus in the case of sets $w(i) = i$, as defined in [2]. For any subset $A = (a_1, a_2, \ldots, a_m)$ of M, let $w(A) = \sum_{i=1}^{m} w(i) a_i$. For any $i \in [m]$, let $N(A, i) = \sum_{j=1}^{i} (c_j - a_j)$ count, with multiplicity, the number of elements in A^c that are less than or equal to i.

For any subset A in $\mathcal{M}_k$, and any element $i \in A^c$, let the color of the edge from A to $A \cup \{i\}$ be $(N(A, i) + w(A)) \bmod (n-k)$. By definition, this is an $(n-k)$-edge-coloring. We show that this forms a proper edge-coloring of the kth level graph.

Suppose for some subset $A \in \mathcal{M}_k$ and distinct elements $i, j \in A^c$ with $i < j$, the edges from A to $A \cup \{i\}$ and $A \cup \{j\}$ get the same color. Then

$$N(A, i) + w(A) \equiv N(A, j) + w(A) \bmod (n-k)$$

which implies $N(A, j) - N(A, i) \equiv 0 \bmod (n-k)$.

By definition, $N(A, j) - N(A, i) = \sum_{l=i+1}^{j} (c_l - a_l)$. Since $j \in A^c$, $a_j < c_j$, which implies $N(A, j) - N(A, i) > 0$. On the other hand, since A has size k, $\sum_{l=1}^{m} (c_l - a_l) = n - k$. Since $i \in A^c$, $a_i < c_i$, which implies $N(A, j) - N(A, i) < n - k$. This contradicts $N(A, j) - N(A, i) \equiv 0 \bmod (n-k)$. Thus all edges incident with any subset in $\mathcal{M}_k$ get distinct colors.

Similarly, suppose there is a set $B = (b_1, b_2, \ldots, b_m)$ in $\mathcal{M}_{k+1}$ and distinct elements $i < j$ in B, such that the edges from $B \setminus \{i\}$ and $B \setminus \{j\}$ to B get the same color. Then

$$N(B \setminus \{i\}, i) + w(B \setminus \{i\}) \equiv N(B \setminus \{j\}, j) + w(B \setminus \{j\}) \bmod (n-k).$$

Since $N(B \setminus \{i\}, i) = N(B, i) + 1$, $N(B \setminus \{j\}, j) = N(B, j) + 1$, $w(B \setminus \{i\}) = w(B) - w(i)$ and $w(B \setminus \{j\}) = w(B) - w(j)$, this implies

$$N(B, i) - w(i) \equiv N(B, j) - w(j) \bmod (n-k)$$

and hence

$$N(B, j) - N(B, i) \equiv w(j) - w(i) \bmod (n-k).$$

By definition, $N(B, j) - N(B, i) = \sum_{l=i+1}^{j} (c_l - b_l)$, and $w(j) - w(i) = \sum_{l=i+1}^{j} c_l$. Therefore, $\sum_{l=i+1}^{j} b_l \equiv 0 \bmod (n-k)$. Since $i, j \in B$, $b_i, b_j > 0$ and $\sum_{l=1}^{m} b_l = k + 1 \leq n - k$. This implies $0 < \sum_{l=i+1}^{j} b_l < n - k$, a contradiction.

This completes the proof that the given edge-coloring is a proper edge-coloring with $n - k$ colors.

3. Lexical matchings

In this section, we generalize the lexical 1-factorization of the middle levels of the Boolean lattice to the middle levels in the multiset lattice. It is not clear if this can be generalized easily to all levels, even for a Boolean lattice.

For any subset $A = (a_1, a_2, \ldots, a_m)$ of M and any $i \in [m]$, let $w(A, i) = a_i + a_{i+1} - c_i$, with $a_{m+1} = a_1$. For any two distinct elements $i, j \in [m]$, let $w(A, i, j) = \sum_{l \in [i,j)} w(A, l)$. Note that $w(A, i, j) + w(A, j, i) = 2 \sum_{l=1}^{m} a_l - \sum_{l=1}^{m} c_l = 2|A| - n$.

Since we are considering only the middle levels, we assume here that n is odd and $k = (n-1)/2$. For a subset A in $\mathcal{M}_{(n-1)/2}$ and any element $i \in [m]$, let $S(A, i) = \{j \mid j \in [m] \setminus \{i\}, w(A, j, i) < 0\}$. For a subset B in $\mathcal{M}_{(n+1)/2}$ and $i \in [m]$, let $S(B, i) = \{j \mid j \in [m] \setminus \{i\}, w(B, i, j) > 0\}$.

Lemma 2. *For any subset A in $\mathcal{M}_{(n-1)/2}$ and any two distinct elements $i, j \in [m]$, if $w(A, i, j) < 0$ then for all $l \in [m] \setminus \{i, j\}$, $w(A, l, j) \leq w(A, l, i)$. Similarly, for any subset B in $\mathcal{M}_{(n+1)/2}$, if $w(B, i, j) > 0$ then $w(B, i, l) \geq w(B, j, l)$ for all $l \in [m] \setminus \{i, j\}$.*

Proof. If $l \notin [i, j)$, then $w(A, l, j) = w(A, l, i) + w(A, i, j) < w(A, l, i)$. Also, $w(B, i, l) = w(B, i, j) + w(B, j, l) > w(B, j, l)$. If $l \in [i, j)$, then $w(A, l, j) = w(A, i, j) - w(A, i, l)$. Since $w(A, i, l) + w(A, l, i) = -1$, $w(A, l, j) = w(A, i, j) + 1 + w(A, l, i) \leq w(A, l, i)$. Similarly, $w(B, i, l) = w(B, i, j) - w(B, l, j)$. Since $|B| = (n+1)/2$, $w(B, j, l) = 1 - w(B, l, j)$, and $w(B, i, l) = w(B, i, j) - 1 + w(B, j, l) \geq w(B, j, l)$. □

Lemma 3. *For any subset A in $\mathcal{M}_{(n-1)/2}$ and any two distinct elements $i, j \in [m]$, either $S(A, i) \subset S(A, j)$ and $i \in S(A, j)$ or $S(A, j) \subset S(A, i)$ and $j \in S(A, i)$.*

Proof. Since $w(A, i, j) + w(A, j, i) = -1$, it follows that either $w(A, i, j) < 0$ or $w(A, j, i) < 0$. Without loss of generality, assume $w(A, i, j) < 0$ and hence $w(A, j, i) \geq 0$. Therefore, $i \in S(A, j)$ and $j \notin S(A, i)$. By definition, $i \notin S(A, i)$ and $j \notin S(A, j)$. For any $l \in [m] \setminus \{i, j\}$, if $l \in S(A, i)$ then $w(A, l, i) < 0$ and Lemma 2 implies $w(A, l, j) < 0$ and $l \in S(A, j)$. Thus $S(A, i) \subset S(A, j)$ and $i \in S(A, j)$. □

Lemma 4. *For any subset B in $\mathcal{M}_{(n+1)/2}$, for any element $i \in B$, and any element $j \in [m] \setminus \{i\}$, $w(B, i, j) = -w(B \setminus \{i\}, j, i)$, which implies $S(B, i) = S(B \setminus \{i\}, i)$. For any two distinct elements $i, j \in [m]$, either $S(B, i) \subset S(B, j)$ and $i \in S(B, j)$ or $S(B, j) \subset S(B, i)$ and $j \in S(B, i)$.*

Proof. Since B has size $(n+1)/2$, $w(B, i, j) + w(B, j, i) = 1$ for all distinct elements $i, j \in [m]$. Also, for any $i \in B$, $w(B \setminus \{i\}, j, i) = w(B, j, i) - 1$, which implies $w(B, i, j) = -w(B \setminus \{i\}, j, i)$. Thus $j \in S(B, i)$ iff $j \in S(B \setminus \{i\}, i)$ and $S(B, i) = S(B \setminus \{i\}, i)$.

Since $w(B, i, j) + w(B, j, i) = 1$, without loss of generality, $w(B, i, j) > 0$ and hence $w(B, j, i) \leq 0$. Then $j \in S(B, i)$ but $i \notin S(B, j)$. Lemma 2 implies that if $l \in S(B, j)$ then $w(B, i, l) \geq w(B, j, l) > 0$, and $l \in S(B, i)$. Therefore $S(B, j) \subset S(B, i)$ and $j \in S(B, i)$. $\square$

We describe an explicit $((n+1)/2)$-edge-coloring of the middle levels graph. Let $A = (a_1, a_2, \ldots, a_m)$ be an arbitrary subset in $\mathcal{M}_{(n-1)/2}$ and $i \in A^c$. Let the color of the edge from A to $A \cup \{i\}$ be

$$1 + \sum_{j \in S(A,i)} \min(-w(A, j, i), a_j).$$

Since $\sum_{j=1}^{m} a_j = (n-1)/2$, it follows that this is a $((n+1)/2)$-edge-coloring.

Suppose for two distinct elements $i, j \in A^c$, the edges from A to $A \cup \{i\}$ and $A \cup \{j\}$ get the same color. Without loss of generality, we may assume $w(A, i, j) < 0$ and hence $S(A, i) \subset S(A, j)$. By Lemma 2, for every element $l \in S(A, i)$, $w(A, l, j) \leq w(A, l, i)$, and hence $\min(-w(A, l, j), a_l) \geq \min(-w(A, l, i), a_l)$. This implies

$$\sum_{l \in S(A,j)} \min(-w(A, l, j), a_l) \geq \sum_{l \in S(A,i)} \min(-w(A, l, i), a_l).$$

For equality to hold,

$$\min(-w(A, l, i), a_l) = \min(-w(A, l, j), a_l) \text{ for all } l \in S(A, i)$$

and $a_l = 0$ for $l \in S(A, j) \setminus S(A, i)$. In particular, $a_i = 0$. We show that this gives a contradiction.

Since $j \in A^c$, $a_j < c_j$, therefore

$$\sum_{l \in [i,j]} (2a_l - c_l) = w(A, i, j) + a_i + a_j - c_j \leq -2.$$

This implies $[i, j] \neq [m]$ and $\sum_{l \in (j,i)} (2a_l - c_l) \geq 1$. Let $p \in (j, i)$ be an element such that $\sum_{l \in [p,i)} (2a_l - c_l) \geq 1$ and $|[p, i)|$ is minimum among all such possibilities. We show that $p \in S(A, j)$ and either $p \notin S(A, i)$ but $a_p > 0$ or $p \in S(A, i)$ but $\min(-w(A, p, j), a_p) > \min(-w(A, p, i), a_p)$, giving a contradiction in either case.

The choice of p implies $\sum_{l\in(p,i)}(2a_l - c_l) \le 0$, which implies

$$w(A,p,i) = \sum_{l\in(p,i)} (2a_l - c_l) + a_p - c_p + a_i \le 0$$

and $w(A,p,j) = w(A,p,i) + w(A,i,j) < 0$. Thus $p \in S(A,j)$. Also,

$$1 \le \sum_{l\in[p,i)} (2a_l - c_l) = w(A,p,i) + a_p.$$

If $w(A,p,i) = 0$, then $p \notin S(A,i)$, but $a_p \ge 1$, a contradiction. If $w(A,p,i) < 0$, then $p \in S(A,i)$, but $a_p > -w(p,A,i)$. Since $w(A,p,j) < w(A,p,i)$ this implies $\min(-w(A,p,j)), a_p) > -w(A,p,i)$, a contradiction. Therefore, no two edges incident with a set A in $\mathcal{M}_{(n-1)/2}$ can have the same color.

Suppose for a set $B = (b_1, b_2, \ldots, b_m)$ in $\mathcal{M}_{(n+1)/2}$ and distinct elements $i, j \in B$, the edges from $B\setminus\{i\}$ and $B\setminus\{j\}$ to B get the same color. By definition, the color of the edge from $B\setminus\{i\}$ to B is $1+\sum_{l\in S(B\setminus\{i\},i)} \min(-w(B\setminus\{i\},l,i), b_l)$, which by Lemma 4 is equal to $1 + \sum_{l\in S(B,i)} \min(w(B,i,l), b_l)$. Similarly, the edge from $B\setminus\{j\}$ to B is colored $1+\sum_{l\in S(B,j)} \min(w(B,j,l), b_l)$. Since $w(B,i,j) + w(B,j,i) = 1$, we may assume, without loss of generality, $w(B,i,j) > 0$ and $w(B,j,i) \le 0$. By Lemma 2, $w(B,i,l) \ge w(B,j,l)$ for all $l \in [m] \setminus \{i,j\}$, and by Lemma 4, $S(B,j) \subset S(B,i)$. Therefore for every $l \in S(B,j)$, $\min(w(B,i,l), b_l) \ge \min(w(B,j,l), b_l)$. But $j \in S(B,i)$ and $b_j > 0$ since $j \in B$. This implies

$$\sum_{l\in S(B,i)} \min(w(B,i,l), b_l) > \sum_{l\in S(B,j)} \min(w(B,j,l), b_l)$$

giving a contradiction. This proves that the edge-coloring defined is a proper $((n+1)/2)$-edge-coloring.

4. Remarks

The original motivation for constructing the 1-factorization of the middle levels was to find two disjoint perfect matchings whose union forms a Hamiltonian cycle. It is now known that the middle levels graph of the Boolean lattice is Hamiltonian [4, 7]. In the case of multisets, if some $c_i > n/2$, then the middle levels graph has a vertex of degree 1, so a Hamiltonian cycle is ruled out. It would be interesting to see if the middle levels graph in the divisor lattice always has a Hamiltonian path, and a Hamiltonian cycle if each $c_i < n/2$ for all i.

Although the edge-colorings constructed here use the minimum number of colors, the colors may be distributed arbitrarily at the vertices. It would be interesting to see if there always exists an edge-coloring of the kth level graph such that for any set $A \in \mathcal{M}_k$ of degree d, the colors of edges incident with A are $1, 2, \ldots, d$. In the Boolean lattice, all sets in $\mathcal{M}_k$ have the same degree, which is also the maximum degree, so this holds. This is not the case for the divisor lattice and it is not obvious such an edge-coloring exists, even when the maximum degree is m.

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