A generalization of Mader's theorem

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Abstract

One of the relatively less known theorems of Mader states that every graph G with minimum degree $d \geq 1$ contains an edge uv such that there are d internally vertex-disjoint paths between u and v in G. We give a generalization of this theorem. Let T be any rooted tree with $d+1$ vertices. There is a rooted subtree T' of G isomorphic to T, such that G contains d paths that pairwise intersect in the root of T' and join the root to the other d vertices in T'. The proof technique is essentially the same as Mader's.

1 Introduction

One of the most basic results in graph theory, attributed to folklore, is that every graph with minimum degree d contains every tree with $d+1$ vertices. Another such result is that every graph with minimum degree $d \geq 2$ contains a cycle of length $\geq d+1$. Brandt [1] showed that every graph of order n and minimum degree d contains every forest with d edges and at most n vertices. An early result of Mader [4] is that every graph with minimum degree $d \geq 1$ contains an edge such that there are d internally vertex-disjoint paths between the endvertices of the edge. A well-known theorem of Dirac [3] is that every graph with minimum degree 3 contains a subdivision of K^4. Thomassen and Toft [5] showed that a graph with minimum degree 3 contains a subdivision of K^4 such that edges in a hamilton path of K^4 are not subdivided.

These results suggest the following general question: What are the graphs that are guaranteed to be "contained" in any graph with minimum degree d? In order to make this precise and meaningful, we need to define what we mean by "contain" more carefully.

Let $\mathcal{T}(H)$ denote the set of all graphs that can be obtained from a graph H by subdividing the edges of H. A graph H is a topological minor of a graph G if there is a subgraph of G that belongs to $\mathcal{T}(H)$ [2]. We write $H \preceq G$ if H is a topological minor of G. If F is a forest in a graph H, we denote by $\mathcal{T}(H, F)$ the set of all graphs that can be obtained by subdividing the edges of H that are not in F. With a slight abuse of notation, we say $(H, F) \preceq G$ if there is a subgraph of G that belongs to $\mathcal{T}(H, F)$.

We can now restate the theorems as follows. Let G be a graph with minimum degree $d \geq 1$. Then

- $(T, T) \preceq G$ for any tree T with $d + 1$ vertices.
- $(C^{d+1}, P^{d+1}) \preceq G$, if $d \geq 2$.
- $(K^1 * K_{1,d-1}, K_{1,d}) \preceq G$ (Mader).
- $K^4 \preceq G$, if $d \geq 3$ (Dirac).
- $(K^4, P^4) \preceq G$, if $d \geq 3$ (Thomassen and Toft).

In this note, we generalize Mader's theorem from this point of view. We will prove that if F is any forest with d vertices, then $(K^1 * F, T) \preceq G$, where T is any spanning tree in $K^1 * F$ that contains F. Another way of viewing this result is that if T is a rooted tree with $d + 1$ vertices, then there is a rooted subtree T' of G isomorphic to T, such that G contains d paths that pairwise intersect in the root of T' and join the root to the other d vertices in T'. Thus if T is $K_{1,d}$ with a leaf vertex as the root, this implies Mader's theorem.

We believe this is not the most general theorem. It should be possible to show that $(K^1 * F, T) \preceq G$ for any spanning tree T in $K^1 * F$. Even more generally, it is possible that $(H, T) \preceq G$, where H is a connected series-parallel graph with $d + 1$ vertices and T a spanning tree in H. Note that if T is a tree with d vertices, then $K^1 * T$ is a series-parallel graph with $d + 1$ vertices. However, Mader's proof technique does not seem to extend to these problems.

2 Main Result

We first describe the terminology used. Most of it is standard and may be found in, for example, [2].

If A and B are disjoint subsets of vertices, an A–B path is a path with one end in A and the other in B and whose internal vertices are not in $A \cup B$. A collection of paths is said to be internally vertex-disjoint if no path contains an internal vertex of any other path. A v–S fan is a collection of $|S|$ v–S paths such that any two paths have only the vertex v in common. If P is a u–v path and x a vertex in P, we denote by $P[u, x]$ ($P[x, v]$) the u–x (x–v) path that is a subpath of P.

Theorem 1 *Let G be a graph with minimum degree $\delta(G) \geq d$ and let T be any rooted tree with $d+1$ vertices. Then there is a subtree T' of G isomorphic to T with root r', such that G contains a r'-$V(T')$ fan.*

In order to prove Theorem 1 by induction, we need to prove something stronger. We introduce some more definitions.

An *ordered clique* K in a graph G is a clique in G with an ordering imposed on its vertices. Let K be an ordered clique with vertices ordered $v_1, v_2, \ldots, v_k$. We say a subtree T' of G is *consistent* with the ordered clique K, if for every vertex $v_i \in V(T') \cap V(K)$, v_i has at most one neighbour in T' that is not contained in $\{v_1, v_2, \ldots, v_{i-1}\}$, for $1 \leq i \leq k$. We denote the degree in T of the root of a rooted tree T by $d(T)$.

Theorem 2 *Let G be a graph and T a rooted tree with $d+1$ vertices. Let K be an ordered clique in G such that every vertex in $G-K$ has degree at least d in G. Suppose $G-K$ contains a vertex of degree at least $d(T)$. Then there is a subtree T' of G isomorphic to T that satisfies the following properties.*

1. *The root r' of T' and the neighbours of r' in T' are contained in $G-K$.*
2. *T' is consistent with the ordered clique K.*
3. *G contains a r'–$V(T')$ fan.*

Proof: Let $v_1, v_2, \ldots, v_k$ be the ordering of the vertices in K. Let L be a maximal clique in G containing K and suppose the vertices of L are ordered $v_1, \ldots, v_k, \ldots, v_l$. Here $v_{k+1}, \ldots, v_l$ are the vertices in $L-K$. We will say that a vertex v_i in L is larger than v_j if $i > j$.

We consider two cases.

Case 1. $G-L$ does not contain a vertex with degree $\geq d(T)$.

Assume the vertices of T are labeled $t_1, t_2, \ldots, t_{d+1}$ in reverse breadth-first order such that t_{d+1} is the root and every t_i has exactly one neighbour t_j with $j > i$, for $1 \leq i < d+1$. We call this unique neighbour the parent of t_i. All other neighbours of t_i are its children. Let $d(T) = t$ and let $d_1, d_2, \ldots, d_t$ be the degrees of the children of the root in T. Let $d' = \sum_{i=1}^{t} d_i \leq d$. Let $L' = L - \{v_1, v_2, \ldots, v_{d-d'}\}$.

Suppose $G-L$ is empty. Since $G-K$ is not empty and every vertex in $G-K$ has degree at least d in G, G must be a clique with at least $d+1$ vertices. Since $G-K$ contains a vertex of degree at least t, $|L-K| \geq t+1$. Let the vertex t_i of T correspond to the vertex $v_{i+l-d-1}$ in G, for $1 \leq i \leq d+1$. This gives an isomorphism from T to a subtree T' of G that satisfies all the properties stated in Theorem 2.

Suppose $G-L$ is not empty. We will first choose the root r' of T' and its children $c_1, c_2, \ldots, c_t$ in $G-K$. Let S denote the set $\{c_1, \ldots, c_t\}$.

If there is a vertex in $G-L$ with degree at least t in $G-K$, choose such a vertex r' with maximum degree in $G-L$ as the root of T'. Let $c_1, c_2, \ldots, c_s$, $s < t$ be neighbours of r' in $G-L$. These will be children of r' in T'. Since the degree of r' is at least d in G, r' has at least $d-s$ neighbours in L and hence at least $d'-s \geq t-s$ neighbours in L'. Let $c_{s+1}, \ldots, c_t$ be the $t-s$ largest neighbours of r' in L', with c_t the smallest of them. These will be the remaining children of r' in T'. Note that since r' has degree at least t in $G-K$, c_t is not in K.

We claim that for $1 \leq i \leq s$, the vertex c_i has at least $d'-t$ neighbours in $V(L') \setminus S$. Since its degree in G is at least d, if it had less than $d'-t$ neighbours in $V(L') \setminus S$, it must have at least $s+1$ neighbours in $G-L$ and at least $t+1$ neighbours in $G-K$. This contradicts the choice of the root r'. Similarly, since r'

has at least $d' - t$ neighbours smaller than c_t in $V(L') \setminus S$, each of the vertices c_i, for $s+1 \leq i \leq t$, has $d' - t$ neighbours smaller than c_t in $V(L') \setminus S$.

If every vertex in $G - L$ has degree less than t in $G - K$, let r' be any vertex in $G - K$ with degree at least t in $G - K$. Let $c_1, c_2, \ldots, c_t$ be any t neighbours of r' in $G - K$. Every vertex in $G - L$ has degree at least d in G and hence has at least $d - t + 1$ neighbours in K. Since $G - L$ is not empty, this implies $|K| \geq d - t + 1$. Therefore all vertices $c_1, c_2, \ldots, c_t$ have at least $d - t + 1$ neighbours in K and hence at least $d' - t + 1$ neighbours in $K - \{v_1, v_2, \ldots, v_{d-d'}\}$. Note that vertices in K are smaller than the vertices in $L - K$.

From the above construction, in both cases, each c_i, for $1 \leq i \leq t$, has at least $d' - t$ neighbours in $V(L') \setminus S$ and if $c_i \in L$ it has at least $d' - t$ neighbours in $V(L') \setminus S$ that are smaller than itself. Since $\sum_{i=1}^{t}(d_i - 1) = d' - t$, we can find disjoint subsets $S_1, S_2, \ldots, S_t$ of vertices in $V(L') \setminus S$ such that $|S_i| = d_i - 1$ and every vertex in S_i is adjacent to c_i, for $1 \leq i \leq t$. Further, if $c_i \in V(L)$, the vertices in S_i are smaller than c_i. Now join every vertex in S_i to c_i in T'. The remaining $d - d'$ vertices of T, $t_1, t_2, \ldots, t_{d-d'}$ are mapped to the vertices $v_1, v_2, \ldots, v_{d-d'}$ respectively, and we add edges joining these to the vertex in T' corresponding to their parent. The construction ensures that T' satisfies the first two properties in Theorem 2.

Let $A = V(T')$ and let B be the set of neighbours of r' in G together with r'. Then $|A| = d+1$ and $|B| \geq d+1$, hence there is an injection f from $A \setminus B$ to $B \setminus A$. For a vertex $v \in A \cap B$, if $v \neq r'$, the edge $r'v$ forms the path from r' to v in the r'–$V(T')$ fan, otherwise the vertex r' by itself forms a path in the fan. In particular, for every child of r', the path in the fan has length one and is an edge in T'. Any vertex $v \in A \setminus B$ is not a child of r' and must be in L. Similarly, a vertex $u \in B \setminus A$ must be in L. Hence $r', f(v), v$ is path from r' to v in the r'–$V(T')$ fan. These paths give the required r'–$V(T')$ fan in G.

This completes the proof of Case 1.

Case 2. $G - L$ contains a vertex of degree $\geq d(T)$.

Since L is a maximal clique, every vertex in $G - L$ is not adjacent to at least one vertex in L. For a vertex v in $G - L$, let $\pi(v)$ denote the largest vertex in L that is not adjacent to v.

Let $G' = (G - v_l) \cup \{v\pi(v) : v \in G - L \text{ and } \pi(v) \neq v_l\}$ and let $L' = L - v_l$ be the ordered clique in G' with the ordering $v_1, v_2, \ldots, v_{l-1}$. Every vertex in $G' - L' = G - L$ has degree at least d in G' and $G' - L'$ contains a vertex of degree at least $d(T)$. By the induction hypothesis, there is a subtree T' of G' isomorphic to T and consistent with the ordered clique L', such that the root r' of T' and its children are in $G' - L'$, and G' contains a r'–$V(T')$ fan. We may assume that for r', the path in the fan has length zero, while for the children of r', the path in the fan has length one. Further, any path in the fan that intersects L' contains at most one edge in L'. Note that for every vertex $v \in V(T') \cap V(L')$, the r'–v path in the fan is internally disjoint from the r'–v path in T'. We will show that we can modify T' to find the required tree in G.

The only edges of G' that are missing in G are edges of the form $v\pi(v)$ for $v \in G' - L'$. If neither T' nor any of the paths in the r'–$V(T')$ fan contain any of these edges, then T' is the required tree in G. Note that T' is consistent with any ordered subclique of L having the same ordering of vertices as L, in particular K.

Suppose that the tree T' and/or the paths in the fan contain some edges of the form $v\pi(v)$ with $v \in G' - L'$. We call these edges *bad* edges.

Suppose there is an edge $v\pi(v)$ in T' incident with the vertex $\pi(v) = v_i$ in L'. Since v and the root of T' are in $G' - L'$, and T' is consistent with L', v must be the parent of v_i and all children of v_i are in L' and are smaller than v_i. Therefore T' cannot contain any other edge of the form $u\pi(u)$ with $\pi(u) = \pi(v) = v_i$. We call any such bad edge in T' a bad *outgoing* edge.

Suppose a path in the r'–$V(T')$ fan contains an edge of the form $v\pi(v)$ with $\pi(v) = v_j$. If $v_j \in V(T')$ then this path must be terminating at v_j. We call these edges bad *incoming* edges. If $v_j \notin V(T')$, we call the bad edge a bad incoming edge if v is nearer to the root than v_j in the path containing this edge, otherwise we say it is a bad outgoing edge.

Summarizing, we note that a vertex v_i in L' satisfies exactly one of the following.

- There are no bad edges incident with v_i.
- There is exactly one bad edge incident with v_i, which may be incoming or outgoing.
- There are exactly two bad edges incident with v_i, one of which is incoming and one outgoing.

Now we describe the transformation that gives the required tree in G. Let $P_1, P_2, \ldots, P_m$ be the paths in the r'–$V(T')$ fan that contain a vertex in L'. Let s_i be the vertex in $P_i \cap L'$ that is nearest to r' in P_i and let t_i be the farthest such vertex. We may assume that either $s_i = t_i$ or $s_i t_i$ is an edge in P_i. Denote by s_i^- the vertex that precedes s_i in P_i. If the vertex t_i is in $V(T')$, the path P_i terminates at t_i, otherwise let t_i^+ be the vertex that succeeds t_i in P_i.

Let $A = \bigcup_{i=1}^m \{s_i\}$ and let $B = \bigcup_{i=1}^m \{t_i\}$. Let $A' \subseteq A$ be the subset of vertices that have a bad incoming edge incident with them, and $B' \subseteq B$ be the subset of vertices that have a bad outgoing edge incident with them. Note that a vertex in $A \setminus B$ ($B \setminus A$) cannot have a bad outgoing (incoming) edge incident with it. Let $A' = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ and let $B' = \{v_{j_1}, v_{j_2}, \ldots, v_{j_q}\}$ such that $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. Let $v_{i_{p+1}} = v_{j_{q+1}} = v_l$. Let $A_1 = (A \cup \{v_l\}) \setminus \{v_{i_1}\}$ if $p \geq 1$ else let $A_1 = A$. Let $B_1 = (B \cup \{v_l\}) \setminus \{v_{j_1}\}$ if $q \geq 1$ else let $B_1 = B$. Note that $|A_1| = |B_1| = m$ and hence there is a bijection f from $B_1 \setminus A_1$ to $A_1 \setminus B_1$.

First delete all edges $s_i t_i$ in paths P_i for which $s_i \neq t_i$. For $1 \leq c \leq p$, replace the bad incoming edge $v_{i_c}^- v_{i_c}$ by the edge $v_{i_c}^- v_{i_{c+1}}$. This procedure replaces the paths $P_i[r', s_i]$ that form an r'–A fan in G' by m paths $P_1', P_2', \ldots, P_m'$ that form an r'–A_1 fan in G.

For $1 \leq c \leq q$, if v_{j_c} is a vertex in T', the bad outgoing edge incident with it must be an edge in T'. Replace the vertex v_{j_c} by the vertex $v_{j_{c+1}}$ in T'. This is possible since $v_{j_{c+1}}$ is adjacent in G to all neighbours of v_{j_c} in T', and since v_{j_c} is smaller than $v_{j_{c+1}}$, the resulting tree is consistent with L. If $v_{j_{c+1}} \in A_1$ then a path P_i' for some $1 \leq i \leq m$ terminates at $v_{j_{c+1}}$. If $v_{j_{c+1}} \notin A_1$ we add an edge joining it to the vertex $f(v_{j_{c+1}}) \in A_1 \setminus B_1$. This gives the path in the r'–$V(T')$ fan in G terminating at $v_{j_{c+1}}$.

If $v_{j_c} \notin V(T')$ then replace the bad outgoing edge $v_{j_c}v_{j_c}^+$ by the edge $v_{j_{c+1}}v_{j_c}^+$. Suppose the vertex v_{j_c} was contained in a path P in the fan joining r' to a vertex $v \in V(T')$. Then $v \notin V(L)$ and we replace the path $P[v_{j_c}, v]$ by the path $v_{j_{c+1}}v_{j_c}^+ \cup P[v_{j_c}^+, v]$. If $v_{j_{c+1}} \in A_1$ then a path P_i' for some $1 \leq i \leq m$ terminates at $v_{j_{c+1}}$. If $v_{j_{c+1}} \notin A_1$ we add the edge joining it to the vertex $f(v_{j_{c+1}}) \in A_1 \setminus B_1$. This gives the path in the r'–$V(T')$ fan in G terminating at v.

This completes the proof of Case 2 and Theorem 2 is proved. □

The proof of Theorem 1 follows easily from Theorem 2. If $d(T) = d$, that is T is $K_{1,d}$ rooted at the center, the theorem is trivial. If $d(T) < d$, we choose K to be a clique containing a single vertex. The hypothesis of Theorem 2 holds and the conclusion follows.

3 Remarks

We mention some corollaries of Theorem 1 that seem interesting by themselves.

Corollary 3 *Let $d = d_1 + d_2 + \cdots + d_k$ where each d_i is a positive integer. Every graph G with minimum degree d contains a vertex v and k neighbours $v_1, v_2, \ldots, v_k$ of v, such that there are d internally vertex-disjoint v-$\{v_1, v_2, \ldots, v_k\}$ paths in G, with exactly d_i paths terminating in v_i.*

Proof: This follows from Theorem 1 by considering T to be rooted tree in which the root has k children and the ith child has degree d_i in T. □

Corollary 3 suggests another possible generalization of Mader's theorem.

Problem 4 *Let T be a weighted tree with k edges $e_1, e_2, \ldots, e_k$ that are assigned positive integer weights $d_1, d_2, \ldots, d_k$ and let G be a graph with minimum degree $d = d_1 + d_2 + \cdots + d_k$. Is there an isomorphism f from T to a subtree T' of G, such that there are d internally vertex-disjoint paths in G, exactly d_i of which join the endpoints of the edge $f(e_i)$ in T'?*

We believe there are even further generalizations possible, especially by considering graphs with more than one connected component.

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