Non-separating Trees in Connected Graphs

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Abstract

Let T be any tree of order $d \geq 1$. We prove that every connected graph G with minimum degree d contains a subtree T' isomorphic to T such that $G - V(T')$ is connected.

1 Introduction.

A subgraph H of a connected graph G is said to be non-separating if $G - V(H)$ is a connected non-empty graph. It is well-known that every non-trivial connected graph G contains a vertex v such that $G - v$ is connected. It is also known that every connected graph with minimum degree two contains a non-separating edge [4] and that every connected graph with minimum degree three contains a non-separating induced cycle [6].

A non-trivial connected graph G is said to be *k-cohesive* if for any two distinct vertices u and v, $d(u) + d(v) + d(u, v) \geq k$, where $d(u)$ is the degree of u and $d(u, v)$ is the distance between u and v. Locke [2] conjectured that for $k \geq 3$, every connected $2k$-cohesive graph contains a non-separating copy of every tree of order k, and proved it for paths [3]. Abreu and Locke [1] proved that every connected $(2k + 2)$-cohesive graph contains a non-separating copy of every tree of order k and diameter at most four.

We show that every connected graph of minimum degree d contains a non-separating copy of every tree of order d. The graph $mK_{d-1} \vee K_1$ for $m \geq 3$, shows that the degree bound is tight for any tree of order d. Our result may be considered to be a partial step towards Locke's conjecture.

The proof is based on a technique used by Mader to prove a completely different result. Mader [5] showed that every graph with minimum degree d contains an edge uv such that there are d internally vertex-disjoint paths between u and v in G. We extend this technique slightly to obtain our result.

The notation used is largely standard and follows, for example, [7]. One difference to be noted is that we will refer to cliques in a graph by K_i, where i is an index and not the order of the clique.

2 Main Result

Theorem 1 *Let G be a connected graph with minimum degree $d \geq 1$. Then for any tree T of order d, G contains a subtree T' isomorphic to T such that $G - V(T')$ is connected.*

Before proving the theorem, we introduce some definitions to describe Mader's technique.

An *ordered clique* K in a graph G is a complete subgraph of G with an ordering imposed on the vertices of K. Let K be an ordered clique in a graph G with the ordering $v_1, v_2, \ldots, v_k$ of its vertices. A subtree T of G is said to be *consistent* with the ordered clique K if every vertex $v_i \in V(K) \cap V(T)$ has at most one neighbor in T that is not contained in $\{v_1, v_2, \ldots, v_{i-1}\}$. We will be considering ordered pairs of the form (G, K), where K is an ordered clique in a graph G.

Definition 2 *Let K be an ordered clique in a graph G and let $v_1, v_2, \ldots, v_k$ be the ordering of the vertices of K. If K is a proper subgraph of G, the reduction $\alpha(G, K)$ of the ordered pair (G, K) is the pair (G', K') defined as follows.*

1. *Suppose there is a vertex $v \in V(G) \setminus V(K)$ such that v is adjacent to all vertices in K. Then let $G' = G$ and $V(K') = V(K) \cup \{v\}$ with the ordering $v_1, v_2, \ldots, v_k, v$ of the vertices of K'. If there is more than one such vertex, any one may be chosen arbitrarily.*

2. *Suppose no vertex in $V(G) \setminus V(K)$ is adjacent to all vertices in K. For each vertex $w \in V(G) \setminus V(K)$, let $\pi(w)$ be the largest index i such that w is not adjacent to $v_i \in V(K)$. Then let $K' = K - v_k$, $V(G') = V(G) \setminus \{v_k\}$ and $E(G') = E(G - v_k) \cup \{wv_{\pi(w)} | w \in N_G(v_k) \setminus V(K)\}$.*

The reduction defined in Definition 2 can be applied repeatedly to an ordered pair (G, K), until $G - V(K)$ is empty. Define $\alpha^0(G, K) = (G, K)$ and $\alpha^i(G, K) = \alpha(\alpha^{i-1}(G, K))$ for $i \geq 1$.

Some obvious properties of this reduction are noted in Lemma 3.

Lemma 3 *Let K be an ordered clique in a graph G and let $(G_i, K_i) = \alpha^i(G, K)$ for some $i \geq 0$. Then the following statements are true.*

1. *$G_i - V(K_i)$ is an induced subgraph of $G - V(K)$.*

2. *The degree in G_i of any vertex in $V(G_i) \setminus V(K_i)$ is equal to its degree in G.*

3. *If $S \subset V(G_i) \setminus V(K_i)$, then $(G_i - S, K_i) = \alpha^i(G - S, K)$.*

Let G be a graph with minimum degree d and let v_1 be any vertex in G. Let $(G_i, K_i) = \alpha^i(G, v_1)$ for $i \geq 0$. Let l be the smallest number such that $G_l - V(K_l)$ contains only one vertex. Since the reduction can be applied until $G_i - V(K_i)$ is empty, and G is non-trivial, there exists such an $l \geq 0$.

Lemma 4 *The vertex v_1 is the first vertex in each of the ordered cliques K_i, for $0 \leq i \leq l$.*

Proof: This is true by definition for $i = 0$. If (G_{i+1}, K_{i+1}) is obtained from (G_i, K_i) by applying step 1 of the reduction, then the first vertex in K_{i+1} is the same as the first vertex in K_i. The same is true if step 2 of the reduction is applied, unless $|K_i| = 1$ and K_{i+1} is empty. However, in this case v_1 has no neighbor in $V(G_i) \setminus V(K_i)$. Hence, by Lemma 3, any component of $G_i - V(K_i)$ is a component of G not containing v_1, contradicting the fact that G is connected. $\square$

Let T be any tree of order d. Let $u_1, u_2, \ldots, u_d$ be an ordering of the vertices of T such that u_i is adjacent to exactly one vertex u_j with $j > i$, for $1 \leq i < d$.

Lemma 5 *There exists a sequence of trees $T_0, T_1, \ldots, T_l$ satisfying the following properties for all $0 \leq i \leq l$.*

1. *T_i is a subtree of G_i isomorphic to T.*
2. *The vertex v_1 is not contained in $V(T_i)$.*
3. *T_i is consistent with K_i.*
4. *Every connected component of $G_i - (V(K_i) \cup V(T_i))$ contains a vertex w such that $|N_{G_i}(w) \cap V(K_i)| > |V(K_i) \cap V(T_i)|$.*

Proof: We construct the sequence $T_0, T_1, \ldots, T_l$ inductively, starting with T_l. Let w be the single vertex in $G_l - V(K_l)$. Since the degree of w is at least d in G_l, $|K_l| \geq d$. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_d}$ be vertices in K_l adjacent to w in G_l such that $i_1 < i_2 < \cdots < i_d$. Now let the vertex u_j of T correspond to the vertex $v_{i_{j+1}}$ in G_l for $1 \leq j < d$, and let the vertex u_d of T correspond to the vertex w in G_l. This gives a subtree T_l of G_l satisfying all properties stated in Lemma 5.

Suppose T_{i+1} is a subtree of G_{i+1} satisfying all properties in Lemma 5 for some $0 \leq i < l$. We show how to construct T_i from T_{i+1}.

Case 1.

Suppose G_{i+1} is obtained from G_i by applying step 1 of the reduction defined in Definition 2. Then $G_i = G_{i+1}$, and $K_i = K_{i+1} - v$, where v is the last vertex in the ordering of $V(K_{i+1})$. Let T_i be the same as T_{i+1}. Then T_i is a subtree of G_i, isomorphic to T and consistent with K_i.

If $v \in V(T_{i+1})$, any component of $G_i - (V(K_i) \cup V(T_i))$ is a component of $G_{i+1} - (V(K_{i+1}) \cup V(T_{i+1}))$. Also, for any vertex w in such a component, $|N_{G_i}(w) \cap V(K_i)| \geq |N_{G_{i+1}}(w) \cap V(K_{i+1})| - 1$. Since $|V(K_i) \cap V(T_i)| = |V(K_{i+1}) \cap V(T_{i+1})| - 1$, by induction, using property 4 in Lemma 5, every such component contains a vertex w with $|N_{G_i}(w) \cap V(K_i)| > |V(K_i) \cap V(T_i)|$.

If $v \notin V(T_{i+1})$, for the component of $G_i - (V(K_i) \cup V(T_i))$ that contains v, $|N_{G_i}(v) \cap V(K_i)| > |V(K_i) \cap V(T_i)|$, since v is adjacent to all vertices in K_i and $v_1 \notin V(T_i)$. Any other component of $G_i - (V(K_i) \cup V(T_i))$ is also a component of $G_{i+1} - (V(K_{i+1}) \cup V(T_{i+1}))$. For any vertex w in such a component, $|N_{G_i}(w) \cap V(K_i)| = |N_{G_{i+1}}(w) \cap V(K_{i+1})|$, hence by induction, using property 4 in Lemma 5, such a component contains a vertex w such that $|N_{G_i}(w) \cap V(K_i)| > |V(K_i) \cap V(T_i)|$.

Case 2.

Suppose G_{i+1} is obtained from G_i by applying step 2 of the reduction. Then $V(G_i) = V(G_{i+1}) \cup \{v_k\}$, $V(K_i) = V(K_{i+1}) \cup \{v_k\}$ and for some subset $X \subseteq V(G_{i+1}) \setminus V(K_{i+1})$, $E(G_i) = (E(G_{i+1}) \setminus \{wv_{\pi(w)} | w \in X\}) \cup \{wv_k | w \in X\}$. Here v_k is the last vertex in the ordering of $V(K_i)$ and $X = N_{G_i}(v_k) \setminus V(K_i)$. We call the edges $\{wv_{\pi(w)}, w \in X\}$ *bad* edges.

If none of the bad edges is contained in T_{i+1} then let T_i be the same as T_{i+1}. Suppose wv_j is a bad edge contained in T_{i+1} for some $w \in X$ and $v_j \in V(K_{i+1})$. Since T_{i+1} is consistent with K_{i+1}, and $v_1 \notin V(T_{i+1})$, all other neighbors of v_j in T_{i+1} are contained in $\{v_2, \ldots, v_{j-1}\}$, hence T_{i+1} can contain at most one bad edge incident with v_j. Similarly, there is at most one bad edge incident with a vertex $w \in X$. Let $w_1v_{j_1}, w_2v_{j_2}, \ldots, w_mv_{j_m}$ be the bad edges contained in T_{i+1}, such that $j_1 < j_2 < \cdots < j_m$. By the definition of the reduction, if wv_p is a bad edge in G_{i+1}, then wv_q is an edge in G_i for all $p < q \leq k$. Construct T_i from T_{i+1} by replacing the vertex $v_{j_p} \in V(T_{i+1})$ by the vertex $v_{j_{p+1}}$ for $1 \leq p \leq m$, where $v_{j_{m+1}}$ is v_k. This is possible since $v_{j_{p+1}}$ is adjacent in G_i to all vertices adjacent to v_{j_p} in T_{i+1}. Since a vertex is replaced by a vertex following it in the ordering, T_i is consistent with K_i and $v_1 \notin V(T_i)$.

If w is a vertex in $V(G_{i+1}) \setminus V(K_{i+1})$, then $|N_{G_i}(w) \cap V(K_i)| = |N_{G_{i+1}}(w) \cap V(K_{i+1})|$ since any bad edge incident with w is replaced by the edge wv_k. Since the connected components of $G_i - (V(K_i) \cup V(T_i))$ are same as the components of $G_{i+1} - (V(K_{i+1}) \cup V(T_{i+1}))$, every such component contains a vertex w such that $|N_{G_i}(w) \cap V(K_i)| > |V(K_i) \cap V(T_{i+1})| = |V(K_i) \cap V(T_i)|$. Hence T_i is the required tree in G_i, satisfying the properties in Lemma 5. □

Theorem 1 now follows from Lemma 5. The tree T_0 obtained in Lemma 5 is a subtree of $G_0 = G$, isomorphic to T and not containing the vertex v_1. Since $V(K_0) = \{v_1\}$, by property 4 in Lemma 5, every connected component of $G - (\{v_1\} \cup V(T_0))$ contains a vertex adjacent to v_1. This implies T_0 is a non-separating subtree of G isomorphic to T. □

References

[1] M. Abreu, S. C. Locke, Non-separating n-trees up to diameter 4 in a $(2n + 2)$-cohesive graph, Proceedings of the Thirty-third Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton,Florida,2002), Congr. Numer. 154 (2002) 21–30.

[2] S. C. Locke, Problem 10647, MAA Monthly 105 (1998) 176.

[3] S. C. Locke, P. Tracy, H.-J. Voss, Highly Cohesive Graphs have Long Non-separating Paths: Solution to Problem 10647, MAA Monthly 108 (2001) 470–472.

[4] L. Lovász, Combinatorial Problems and Exercises, North Holland, 1979, p. 40, Exercise 6.6(b).

[5] W. Mader, Existenz gewisser Konfigurationen in n-gesättigten Graphen und in Graphen genügend großer Kantendichte, Math. Ann. 194 (1974) 295–312.

[6] C. Thomassen, B. Toft, Non-separating induced cycles in graphs, J. Combin. Theory Ser. B 31 (1981) 199–224.

[7] D. B. West, Introduction to Graph Theory, second ed., Prentice-Hall, 2001.