# CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 05, 2025

## Lecture 03 – Theorems, types of proofs

# Logistics and recap

Course material, references are being posted at

- `http://www.cse.iitb.ac.in/~akshayss/teaching.html`

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## Recap of two lectures

- What are discrete structures?
- Course outline
- Chapter 1: Proofs and structures
  - Propositions: statements that can be assigned a truth value
  - Predicates: propositions with variables
  - Quantifiers

# Theorems and proofs

A theorem is a proposition which can be shown true

Classwork: Prove the following theorems.

1. $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$
2. For all $a, b, c \in \mathbb{R}^{\geq 0}$, if $a^2 + b^2 = c^2$, then $a + b \geq c$.
3. If 6 is prime, then $6^2 = 30$.
4. For all $x \in \mathbb{Z}$, $x$ is an even iff $x + x^2 - x^3$ is even.

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5. There are infinitely many prime numbers.
6. There exist irrational numbers $x, y$ such that $x^y$ is rational.
7. For all integers $n > 1$, $n! < n^n$.
8. There does not exist a program which will always determine whether an arbitrary (input-free) program will terminate.

# Theorems and proofs

## Recall: Contrapositive and converse

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To show "$A$ iff $B$", you must show two things:

1. $A$ implies $B$ and
2. its converse, $B$ implies $A$ OR $\neg A$ implies $\neg B$.

# Proof of Theorem 4

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5. Now take any prime $p$ in this product, then, $p$ divides $k$. So, by 3. above, $p \notin \{p_1, \ldots, p_r\}$.
6. This contradicts 1. since we had assumed that $\{p_1, \ldots p_r\}$ was the set of all primes.

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- Case 1: If $z$ is rational, we are done (why?)

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  - Thus we have found two irrationals $x = z = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$ such that $x^y = 2$ is rational. □

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Indeed, note that the above proof is not constructive!

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(H.W): Post a constructive proof of this theorem on piazza.

# Types of proofs

1. $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$
2. For all $a, b, c \in \mathbb{R}^{\geq 0}$, if $a^2 + b^2 = c^2$, then $a + b \geq c$.
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4. $x$ is an even integer iff $x + x^2 - x^3$ is even.
5. There are infinitely many prime numbers.
6. There exist irrational numbers $x, y$ such that $x^y$ is rational.
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   – Direct proof

3. If 6 is prime, then $6^2 = 30$. – Vacuous/trivial proof

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7. For all integers $n > 1$, $n! < n^n$. – next!

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  - if $p$ is true, then $p \vee q$ is true.
  - if $p$ is true and $q$ is true, then $p \wedge q$ is true.
  - if $p$ implies $q$ and $q$ implies $r$, then $p$ implies $r$.
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  - Role of counter-examples: Prove or disprove: For all $x \in \mathbb{N}$, $x^2 + x + 41$ is prime.

# Theorems and proofs

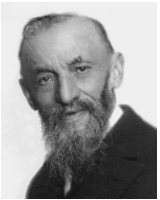
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- Axioms: Peano's axioms, Euclid's axioms.

# Axioms



(a) Euclid



(b) G. Peano





(c) Zermelo-Fraenkel

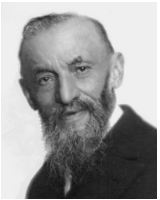
(a) Euclid's axioms for geometry in 300 BCE.

(b) Peano's axioms for natural numbers in 1889.

# Axioms



(a) Euclid



(b) G. Peano





(c) Zermelo-Fraenkel

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(c) Zermelo-Fraenkel and Choice axioms (ZFC) are a small set of axioms from which most of mathematics can be inferred.

- But proving even 2+2=4 requires $>$ 20000 lines of proof!

- In this course, we will assume axioms, mostly from high school math (distributivity of numbers etc.).

# Introducing the world of Mathematical Induction

## Induction (Axiom)

Let $P(n)$ be a property of non-negative integers. If

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- for all $k \geq 0$, $P(k) \implies P(k+1)$ (Induction Step)

then $P(n)$ is true for all $n \in \mathbb{N}$.

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   $(k+1)! = k! \cdot (k+1) \leq k^k(k+1)$ (by Induction Hypothesis)

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$$\begin{aligned}(k+1)! = k! \cdot (k+1) &\leq k^k(k+1) \text{ (by Induction Hypothesis)} \\ &< (k+1)^k \cdot (k+1) = (k+1)^{(k+1)}\end{aligned}$$

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4. Hence by induction, we conclude that for all $n \geq 2$, $n! < n^n$.

# Examples by induction (H.W)

1. Summations:
   1.1 $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
   1.2 $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$

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2. Inequalities
   - 2.1 If $h > -1$, then $1 + nh \leq (1 + h)^n$ for all non-negative integers $n$.
3. Divisibility
   - 3.1 6 divides $n^3 - n$ when $n$ is a non-negative integer.
   - 3.2 21 divides $4^{n+1} + 5^{2n-1}$ whenever $n$ is positive integer.
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– "Proof technique" rather than a "Solution technique" as it requires a good guess of the answer.

# Interesting fallacy in using induction!

Conjecture: All horses have the same colour.

"Proof" by induction on number of horses:

1. Base Case ($n = 1$) The case with one horse is trivial.
2. Induction Hypothesis Assume for $n = k \geq 1$, i.e., any set of $k(\geq 1)$ horses has same color.
3. Induction Step We want to show any set of $k + 1$ horses have same color. Consider such a set, say $1, \ldots, k + 1$.
   (A) First, consider horses $1, \ldots, k$. By induction hypothesis, they have same color.
   (B) Next, consider horses $2, \ldots, k + 1$. By induction hypothesis, they have same color.
   (C) Therefore, 1 has same color as 2 (by A) and 2 has same color as $k + 1$ (by B), implies all $k + 1$ have same color.
4. Thus, by induction, we conclude that for all $n \geq 1$, any set of $n$ horses has the same color. □

Where is the bug?