CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 19, 2025

Lecture 06 – Basic structures: sets and functions

Recap of last 5 lectures

Week 01 and Week 02

- What are discrete structures?
- Chapter 1: Proofs and mathematical reasoning
  - Propositions: statements that can be assigned a truth value
  - Predicates and Quantifiers
  - Theorems and types of proofs: contradiction, contrapositive

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Week 02 and Week 03

- Induction
- Well-ordering principle
- Strong Induction

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- Well-ordering principle
- Strong Induction

Three problem-sheets released on Piazza

1. Questions on Basic proofs, reasoning
2. Questions on Induction, WOP, Strong Induction.

Logistics

Optional Problem Solving Sessions started...

- From now on at 3.30pm Wednesdays (Same venue: CC101/103/105)
- Tomorrow, pop-quiz answersheets will be shown in first 15min.
- Check Piazza before coming!

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Quiz 1: Aug 28th, 3.45pm

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Pop Quiz Anytime!

Quiz 1: Aug 28th, 3.45pm **NOTE THE CHANGE!**

From proofs to structures

From proofs to structures

Next: Chapter 2: Basic Discrete Structures

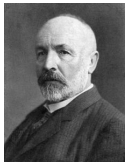
- Finite and Infinite Sets,
- Functions
- Relations

Sets

What is a set?

- A set is an unordered collection of objects.
- The objects in a set are called its elements.

Sets



§ 1

The Conception of Power or Cardinal Number

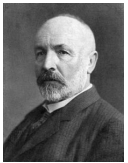
By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Figure: Georg Cantor (1845-1918); extract

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- A set is an unordered collection of objects.
- The objects in a set are called its elements.

More formally,

Let P be a property. Any collection of objects that are defined by (or satisfy) P is a set, i.e., $S = \{x \mid P(x)\}$.

Some simple boring stuff about sets

Examples and properties

- We have already seen examples: $\mathbb{Z}, \mathbb{N}, \mathbb{R}$, set of all horses,...

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So, what is the difference between $\{\emptyset\}$ and $\emptyset$?

Not so simple...

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How do you resolve this?

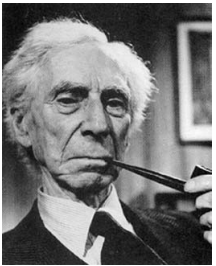


Figure: Bertrand Russell (1872-1970)

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Axiomatic approach to set theory (ZFC!)

Start with a few objects defined. Then for a set A and a property P,
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Why does this definition get rid of Russell's paradox?

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- if $(S \in S)$: from the definition of S, $S \in A$ and $S \notin S$, which is a contradiction.
- if $(S \notin S)$: from the definition, either $S \notin A$ or $S \in S$. But we have assumed that $S \notin S$. Hence, $S \notin A$. No contradiction!

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- Example: {set of all even natural numbers} vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$
- Turns out we need functions... but first...

Hilbert's hotel



- Suppose there is a hotel with infinitely many rooms.
- And suppose they are all full (like in IIT guest house).

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2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

Functions

What you did above was to define functions...

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Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

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(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

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- We write $f(a) = b$ and call b the image of a.
- $Range(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$, $Domain(f) = A$

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Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.

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- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.
- Defined only if $Range(g) \subseteq Domain(f)$.
- Exercise: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?

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Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.
- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

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Composition of functions is associative

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- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

Check it! (H.W.)

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Inverse of a function

- If $f : A \to B$ is a ??? function, then $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$, is called its inverse.

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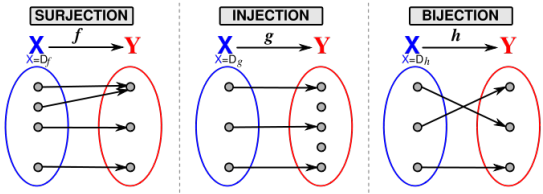
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- If f is a bijection, then $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$

Comparing (finite and infinite) sets

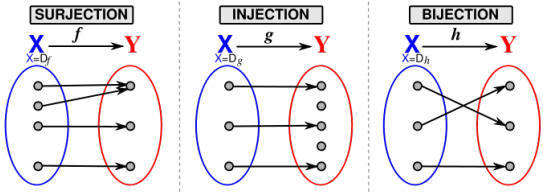


- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.

- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

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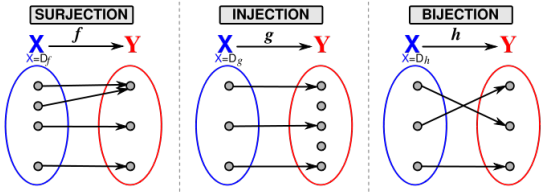


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1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
  – If A, B finite, $|A| \geq |B|$
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
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Properties of finite and infinite sets

Relative notion of "size"

Two finite/infinite sets have the same "size" iff there is a bijection between them.

Properties of finite and infinite sets

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **inj** from A to B and $\exists$ **inj** B to A, implies $\exists$ **bij** from A to B.

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Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.

Properties of finite and infinite sets

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- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- (Schröder-Bernstein Theorem:) $\exists$ **inj** from A to B and $\exists$ **inj** B to A, implies $\exists$ **bij** from A to B. (Read this proof!)

Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.
- What about infinite sets?

Difference between finite and infinite sets

Theorem

For set A, let $b \notin A$. Then A is infinite iff there is a bijection from A to $A \cup \{b\}$.