CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 22, 2025

Lecture 07 – Basic structures: sets and functions

Logistics

Quiz 1: Aug 28th, 3.45pm

Venue: LH classrooms (see Piazza next week for precise seating arrangement!)

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Safe and Attendance

- Any remaining issues: Private message on Piazza to Instructors with proof (screenshots etc).

From proofs to structures

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Chapter 2: Basic Discrete Structures

- Finite sets and Russell's Paradox
- Infinite Sets and comparing them using functions

Hilbert's hotel



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- And suppose they are all full (like in IIT guest house).

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2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

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Let A be a set and $b \notin A$. A is infinite iff there is a bijection from A to $A \cup \{b\}$.

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1. Show contrapositive! If A is finite, then there can't be a bijection from A to $A \cup \{b\}$.

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4. Collecting all such $a_i's$, we get a subset $A' = \{a_i \in A \mid i \in \mathbb{N}\} \subseteq A$. (Note it may be that $A \neq A'$).
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- Even if A, B are infinite, $A \subsetneq B$, there can be a bijection from A to B, i.e., they have the same "cardinality".
- From any set A, there is a surjection from A to $\mathbb{N}$.

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Some questions...

1. Is there a bijection between $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
2. Is there a bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
3. Is there a bijection from $\mathbb{Q}$ to $\mathbb{N}$?
4. Is there a bijection from the set of all subsets of $\mathbb{N}$ to $\mathbb{N}$?
5. Is there a bijection from $\mathbb{R}$ to $\mathbb{N}$?

Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

Countably infinite sets are the "smallest" infinite sets.

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What are the other properties of countable sets?

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Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
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To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.

Unions of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

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- Is this correct?

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Hint: Show that $f(a, b) = \begin{cases} a/b \text{ if } b \neq 0 \\ 0 \text{ if } b = 0 \end{cases}$ is a surjection. How does result follow?

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