CS 105: Department Introductory Course on Discrete Structures

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Lecture 09 – Basic structures: Relations

Summary and moving on...

Week 01, 02 and 03: Proofs and Reasoning

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- Propositions, Predicates, Quantifiers
- Theorems and Types of Proofs
- Induction and variants

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- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Next: Basic Mathematical Structures – Relations

Relations

Definition: Function

Let A, B be two sets. A function f from A to B is a subset R of $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

- Now, suppose A is the set of all Btech students and B is the set of all courses. Clearly, we can assign to each student the courses he/she is taking. Is this a function?

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Definition: Relation

- A relation R from A to B is a subset of $A \times B$.If $(a, b) \in R$, we also write this as aRb.
- Thus, a relation is a way to relate the elements of two (not necessarily different) sets.

Examples and representations of relations

We write $R(A, B)$ for a relation from A to B and just $R(A)$ if $A = B$. Also if A is clear from context, we just write R.

Examples of relations

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- $R_2(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- Let S be a set, $R_3(\mathcal{P}(S)) = \{(A, B) \mid A, B \subseteq S, A \subseteq B\}$.
- Relational databases are practical examples.

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Representations of a relation from A to B.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$.
- As a directed graph.
- As a (database) table.

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Practical application in relational databases: IMDB, university records, etc.

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- Are there other special relations?What are they useful for?

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But, why study relations in this course?

- Functions were special kinds of relations that were useful to compare sets.
- Are there other special relations?What are they useful for?
  - Equivalence relations
  - Partial orders

Partitions of a set – grouping "like" elements

Examples

- Natural numbers are partitioned into even and odd.
- This class is partitioned into sets of students from same hostel.

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- $\bigcup_{S' \in P} S' = S$: its union covers entire set S.
- If $S_1, S_2 \in P$, then $S_1 \cap S_2 = \emptyset$: sets are disjoint.

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Can you think of two trivial partitions that any (non-empty) set must have? (They may coincide for singleton sets)

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Relation generated by a partition

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What properties does this relation have?

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A relation which satisfies all these three properties is called an equivalence relation.

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Definition

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Thus, from any partition, we get an equivalence relation.

Examples

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.
- Equivalence: Reflexive, Symmetric and Transitive.

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Relation	Refl.	Sym.	Trans.	Equiv.
aR_4b if students a and b take same set of courses	✓	✓	✓	✓
aR_5b if student a takes course b				
$\{(a,b) \mid a,b \in \mathbb{Z}, (a-b) \mod 2 = 0\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z} \setminus \{0\}, a \mid b\}$				
$\{(a,b) \mid a,b \in \mathbb{R}, \lvert a-b \rvert < 1\}$				
$\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$				

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Example: natural numbers partitioned into even and odd...

Theorem

Every partition of set S gives rise to a canonical equivalence relation R on S, namely,

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Is the converse true? Can we generate a partition from every equivalence relation?

Equivalence classes

Definition

- Let R be an equivalence relation on set S, and let $a \in S$.
- Then the equivalence class of a, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{b \in S \mid (a, b) \in R\} = \{b \in S \mid aRb\}$.

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In $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid (a - b) \mod 5 = 0\}$, what are $[0]$, $[1]$?

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Lemma

Let R be an equivalence relation on S. Let $a, b \in S$. Then, the following statements are equivalent:

1. aRb
2. $[a] = [b]$
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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Redo the proof formally.)

From equivalence relations to partitions

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Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.
(H.W.): Write the formal proofs of (1) and (2).

More “applications” of equivalence relations

Defining new objects using equivalence relations

Consider $R = \{((a, b), (c, d)) \mid (a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$.

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Defining new objects using equivalence relations

Consider $R = \{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$.

- Then the equivalence classes of R define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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Can we define integers and real numbers starting from naturals by using equivalence classes?

Geometrical objects using equivalence relations

Cut-and-paste

Consider the relation $R([0,1]) = \{aRb \mid a, b \in [0,1]$, either $a = b$ or $a = 1, b = 0$, or $a = 0, b = 1\}$.

- Is R an equivalence relation? What does it define?

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- Is R an equivalence relation? What does it define?
- This is $[0,1]$ in which the end-points have been related to each other.
- So the equivalence classes form a "loop", since end-points are joined. If we imagine $[0,1]$ as a 1-length string, we have glued its ends!

Geometrical objects using equivalence relations

Forming 2D objects

Consider a rectangular piece of the real plane, $[0,1] \times [0,1]$.

- Define $R_1([0,1] \times [0,1])$ by $(a,b)R_1(c,d)$ if
  - $(a,b) = (c,d)$ or
  - $b = d$, $a = 0$, $c = 1$ or
  - $b = d$, $c = 0$, $a = 1$.

  Is R_1 an equivalence relation? What do its equivalence classes define?

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  Is R_1 an equivalence relation? What do its equivalence classes define?
- Define $R_2([0,1] \times [0,1])$ by $(a,b)R_2(c,d)$ if
  - $(a,b) = (c,d)$ or
  - $a, b, c, d \in \{0,1\}$.

  Is R_2 an equivalence relation? What does it define?

Can you build even more interesting "shapes"? Torus? Mobius strip?!