CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Sep 02, 2025

Lecture 10 – Basic structures: relations: posets

Recap and moving on...

Week 01, 02, 03: Proofs and Reasoning

- Propositions, Predicates, Quantifiers
- Theorems and Types of Proofs
- Induction and variants

Week 03, 04, 05: Basic Mathematical Structures

- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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5. Hence both cases lead to contradiction, so our premise must be false, i.e., there is no such bijection.

Last class: Relations

Basic Mathematical Structures – Relations

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- Equivalence relations and partitions of a set

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Basic Mathematical Structures – Relations

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- Equivalence relations and partitions of a set
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- Application: Using equivalence relations to define new mathematical objections

Moving on another special relation: Partial Orders

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Examples:

- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- $R_2(\mathcal{P}(S)) = \{(A, B) \mid A, B \in \mathcal{P}(S), A \subseteq B\}$.

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Definition

A partial order is a relation which is reflexive, transitive and anti-symmetric.

Partial orders and equivalences relations

- Reflexive: $\forall a \in S,\ aRa$.
- Symmetric: $\forall a, b \in S,\ aRb$ implies bRa.
- Anti-symmetric: $\forall a, b \in S,\ aRb, bRa$ implies $a = b$.
- Transitive: $\forall a, b, c \in S,\ aRb, bRc$ implies aRc.

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	Reflexive	Transitive	Symmetric	Anti-symmetric
Equivalence relation	✓	✓	✓	
Partial order	✓	✓		✓

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$\{(a, b) \mid a, b \in \mathbb{Z}^+, a \mid b\}$				
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- Qn: Can a relation be symmetric and anti-symmetric?
- Qn: Can a relation be neither symmetric nor anti-symmetric?

Partially ordered sets (Posets)

Definition

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Examples

- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, |\ )$: positive integers with divisibility relation.

Graphical representation of relations: posets

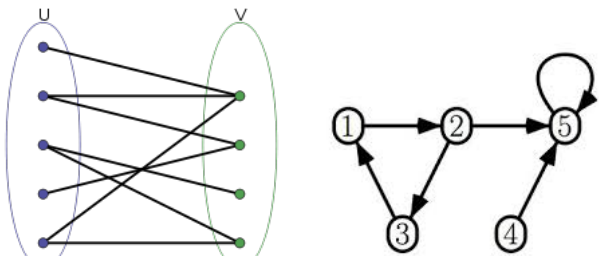
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- nodes as elements of the set and
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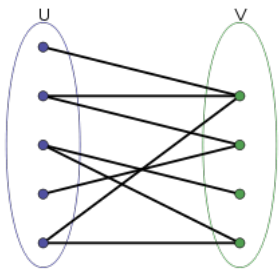
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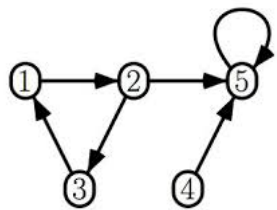


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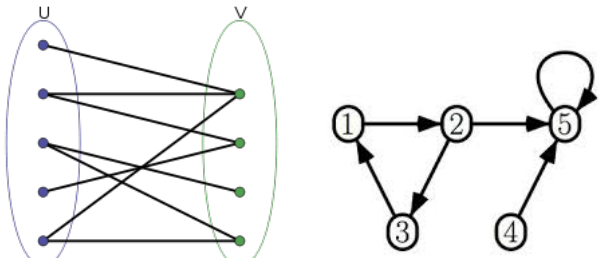


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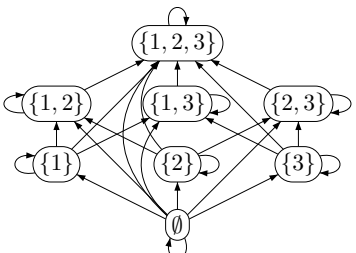
- Did these come from posets?
- Do graphs defined by posets have any "special" properties?

Graphical representation of relations: posets

- Let $S = \{1, 2, 3\}$. Recall the poset $(\mathcal{P}(S), \subseteq)$.
- How does the graph of $(\mathcal{P}(S), \subseteq)$ look like?

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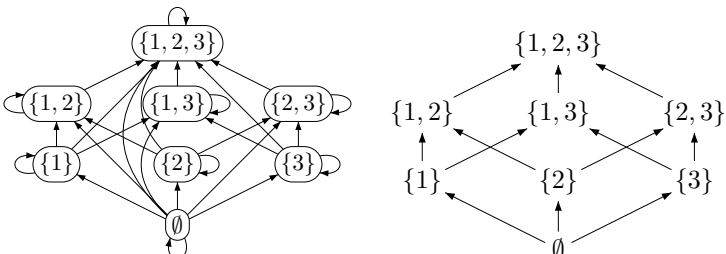


Figure: Graph of a poset and its Hasse diagram

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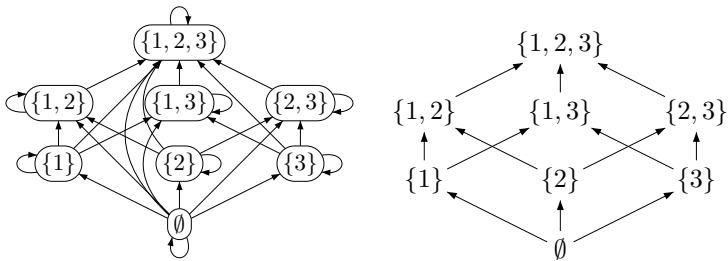


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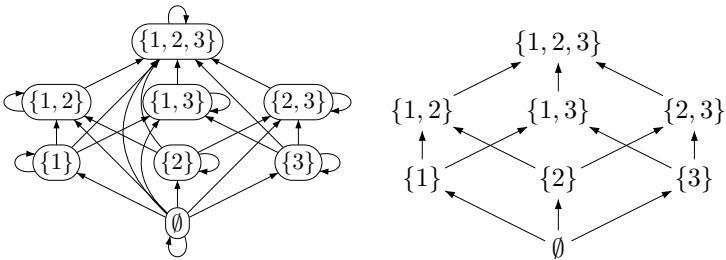


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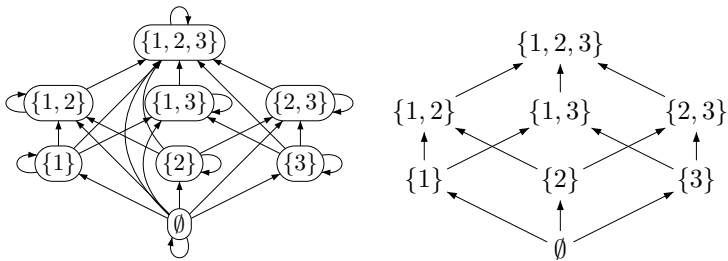


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- Graphs of posets are "acyclic" (except for self-loops).
- Starting from a node and following the directed edges (except self-loops), one can't come back to the same node.
- The reflexive transitive closure of Hasse diagram gives back graph of the poset.

Chains and Anti-chains

Definition

Let $(S, \preceq)$ be a poset. A subset $B \subseteq S$ is called

- a chain if every pair of elements in B is related by $\preceq$.
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Chains and Anti-chains: examples

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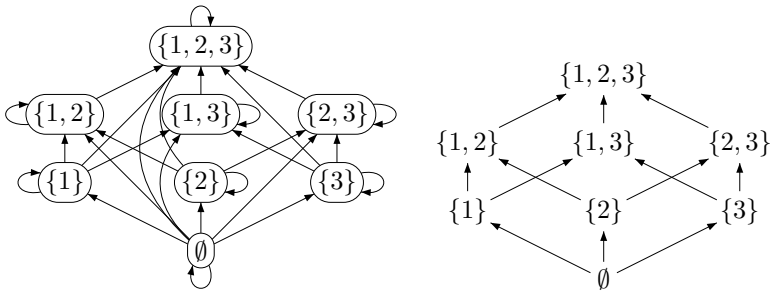


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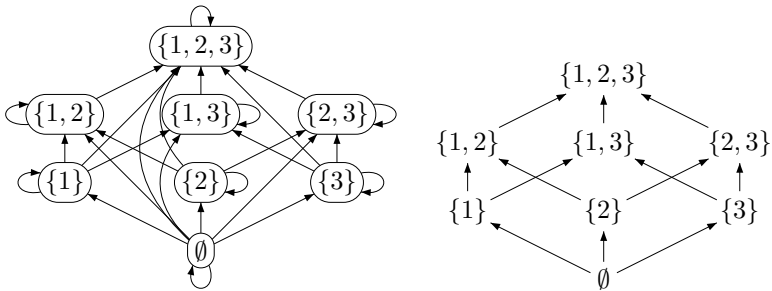


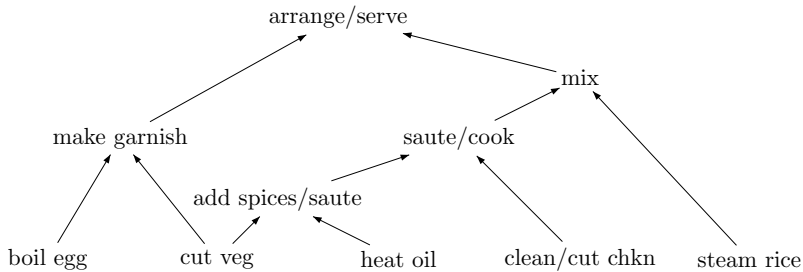
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Examples and applications

A task scheduling example

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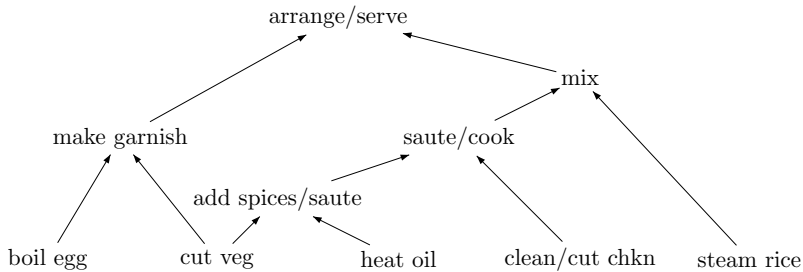


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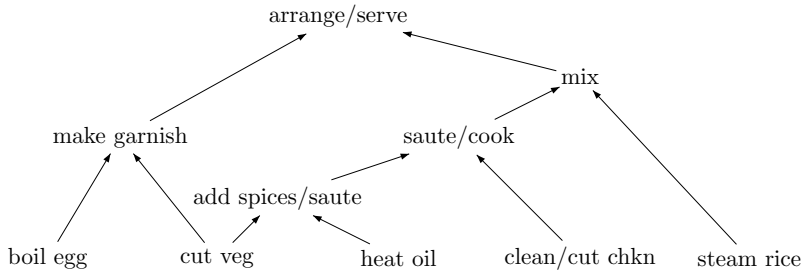


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- Clearly, this shows the dependencies.
- But when you cook you need a total order, right?
- Further, this total order must be consistent with the po.
- This is called a linearization or a topological sorting.

Topological sorting

Definition

A topological sort or a linearization of a poset $(S, \preceq)$ is a poset $(S, \preceq_t)$ with a total order $\preceq_t$ such that $x \preceq y$ implies $x \preceq_t y$.

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Theorem

Every finite poset has a topological sort.

Topological sorting

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A topological sort or a linearization of a poset $(S, \preceq)$ is a poset $(S, \preceq_t)$ with a total order $\preceq_t$ such that $x \preceq y$ implies $x \preceq_t y$.

Theorem

Every finite poset has a topological sort.

Proof: (H.W)

- First prove the following lemma:
  - Every finite non-empty poset has at least one minimal element (x is minimal if $\nexists y, y \preceq x$).

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- Then, construct the chain to complete the proof.