CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Sep 09, 2025

Lecture 11 – Basic structures: chains and anti-chains

Logistics

Optional help session tomorrow: at 3.30pm

- Problemsheet 6

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Harder questions on piazza?

Chains and Anti-chains

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- That is, $\forall a, b \in B$, we have $a \preceq b$ or $b \preceq a$ (or both).

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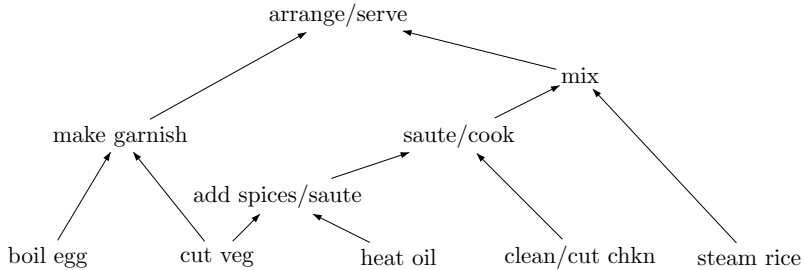
Let $(S, \preceq)$ be a poset. A subset $A \subseteq S$ is called

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- That is, $\forall a, b \in A, a \neq b$, we have neither $a \preceq b$ nor $b \preceq a$.

Examples and applications

A task scheduling example

Let us represent a recipe for making Chicken Biriyani as a poset!

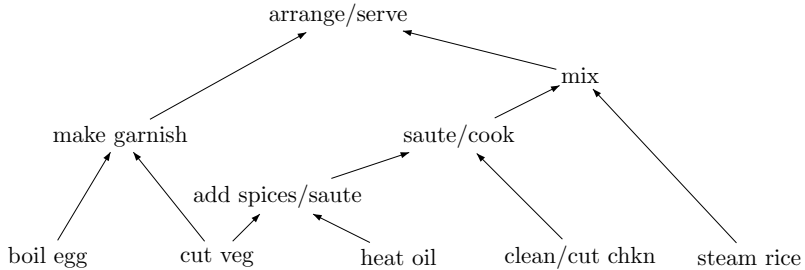


- Dependencies give rise to a poset

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- Dependencies give rise to a poset
- But for cooking you need a total order, that is consistent with the po.
- This is called a linearization or a topological sorting.

Topological sorting

Definition

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Theorem

Every non-empty finite poset has a topological sort.

Minimal (and maximal?) elements

We call x as minimal in S if there exists no $y \in S, y \prec x$ (where, $y \prec x$ is shorthand for $y \preceq x, y \neq x$)

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Proof by induction?(H.W)

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What about infinite posets?

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  - Then consider $S \setminus \{x_1, \ldots, x_k\}$. Call its minimal element x_{k+1}.

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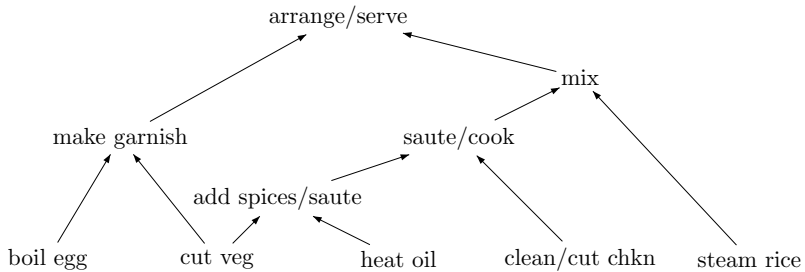
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  - Thus, after n steps we get $x_1 \preceq_t \ldots \preceq_t x_n$ is a chain that is consistent with $\preceq$. □

Topological sort of Biriyani Recipe

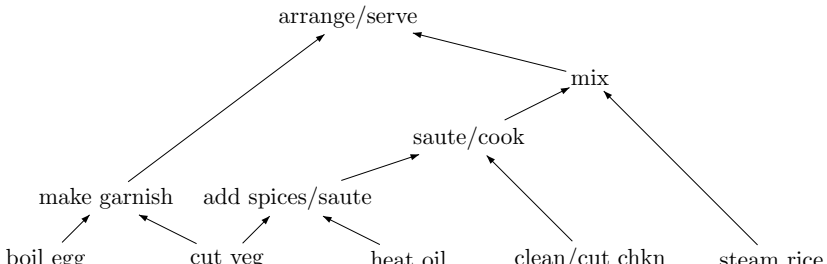
Run the construction on this poset:



Parallel Task Scheduling and chains

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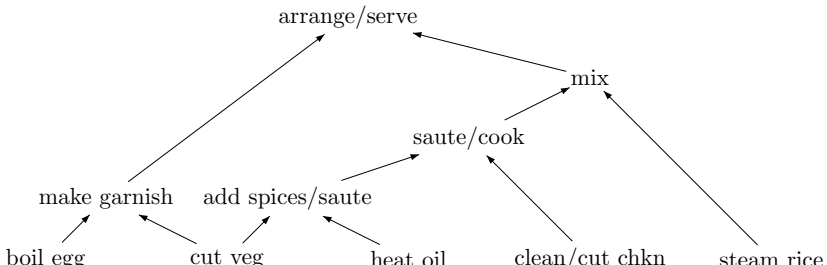
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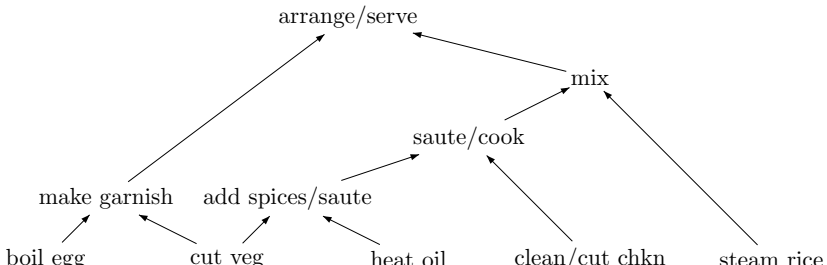


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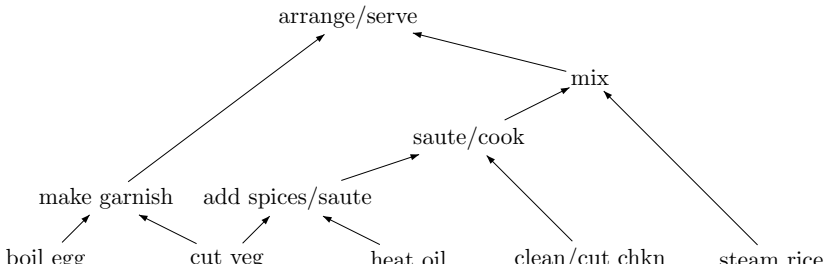


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- How do we schedule the tasks to minimize time used?



- Assume that every task takes 1 time unit.
- Clearly, we still need at least 5 time units.
- That is, the size of the largest/longest chain (size of chain = no. of elements in it).

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Assuming this theorem,

- Observe that we can schedule all of S_i at time i (since we know that all previous tasks were done earlier!).
- Thus, each S_i is an anti-chain.
- This solves the parallel task scheduling problem.

Parallel task scheduling and chains

Theorem

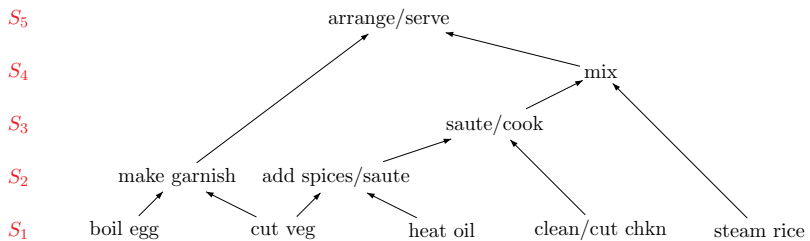
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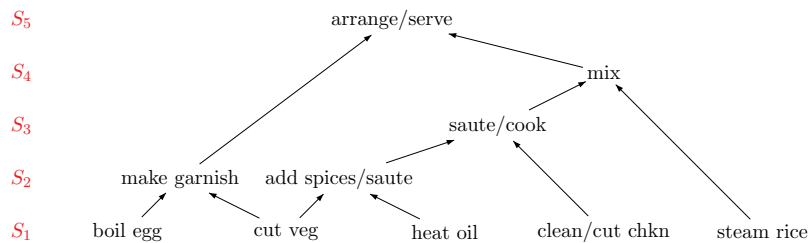


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Proof: Put each $a \in S$ in S_i such that i is the size of the largest chain ending at a. Now proof by contradiction:

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Parallel task scheduling and chains

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- But now, $b \preceq a, b \neq a$ implies we can extend the chain to chain of size $\geq i + 1$, ending at a.
- But then a cannot be in S_i. Contradiction. $\square$

Consequences for chains and anti-chains

Since each S_i was an anti-chain, a celebrated result follows...

Corollary (Mirsky's theorem, 1971)

If the largest chain in a poset $(S, \leq)$ is of size t, then S can be partitioned into t anti-chains.

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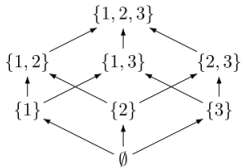
Another corollary (Dilworth's Lemma)

For all $t > 0$, any poset with n elements must have

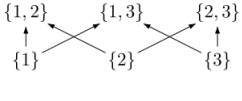
- either a chain of size greater than t
- or an antichain with at least $\frac{n}{t}$ elements.

Exercise: Prove it!

Minimal and maximal elements



Poset $P_1 = (S_1, \subseteq)$
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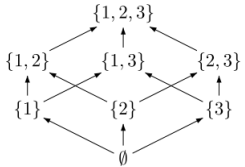


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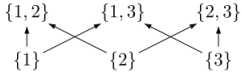
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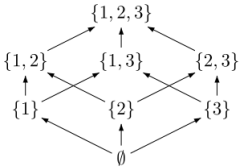


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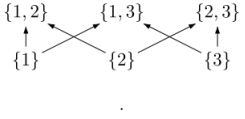
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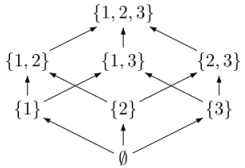
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Exercise: What are the maximal/maximum elements in P_1, P_2?

Least upper bounds and greatest lower bounds

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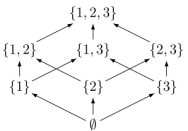
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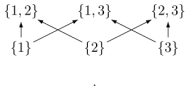
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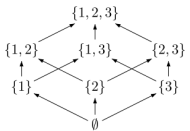


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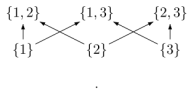
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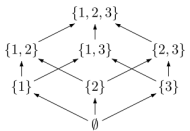
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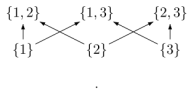
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Some Obervations (Exercise: Prove it!)

- The lub/glb of a subset A in S, if it exists, is unique.
- If the lub/glb of $A \subseteq S$ belongs to A, then it is the greatest/least element of A.

Lattices

Definition

- A lattice is a poset in which every pair of elements has both a lub and a glb (in the set), i.e., $\forall x, y \in S$, there exists $l, u \in S$ such that l is the glb and u is the lub of $\{x, y\}$.

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- Several other applications in many domains of mathematics and CS, including formal semantics of programming languages, program verification.