CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Sep 23, 2025

Lecture 13 – Counting and Combinatorics

Summary till now

Course Outline

1. Proofs and reasoning
   - Propositions, predicates
   - Proofs and proof techniques: contradiction, contrapositive, (strong) induction, well-ordering principle, diagonalization.
2. Basic discrete structures
   - (finite and infinite) sets
   - Functions: injections, surjections, bijections
   - Relations: equivalence relations, partial orders, lattices
   - Some applications
3. Counting and combinatorics
4. Introduction to graph theory

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     - Functions: To compare infinite sets
     - Using diagonalization to prove impossibility results.
     - Equivalences: Defining “like” partitions.
     - Posets: Topological sort, (parallel) task scheduling, lattices
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New Chapter: Counting and Combinatorics

Topics to be covered

- Last lecture before midsem: Basics of counting
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- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

An interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

What will you count here?

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6. Thus number of i such that m_i is odd is even! $\square$

Moving on: Binomial theorem

Recall: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

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Binomial Theorem

Let x, y be variables and $n \in \mathbb{Z}^{\geq 0}$. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$(x+y)^1 = x+y$

$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

...

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(H.W-1) Prove this by induction.

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1. Consider any term $x^i y^j$, where $i + j = n$.
2. To get $x^i y^j$ term in

$$(x+y)(x+y)\cdots(x+y) \qquad (n \text{ times})$$

we need to pick j $y's$ from n sums and remaining x's.

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3. Thus, the coefficient of this term = number of ways to get this term = number of ways to pick j y's from n elts = $\binom{n}{j}$.

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Corollaries

1. $\binom{n}{j} = \binom{n}{n-j}$,
2. $\displaystyle\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$.
3. No. of subsets of n-element set having even cardinality = No. of subsets of n-element set having odd cardinality (=?)

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Corollaries Ex: Using Binomial Thm show the following

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Pascal's Triangle

A recursive way to compute binomial coefficients

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$$\binom{0}{0}$$

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$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

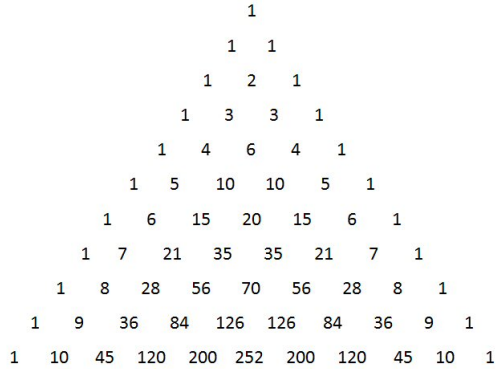
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$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

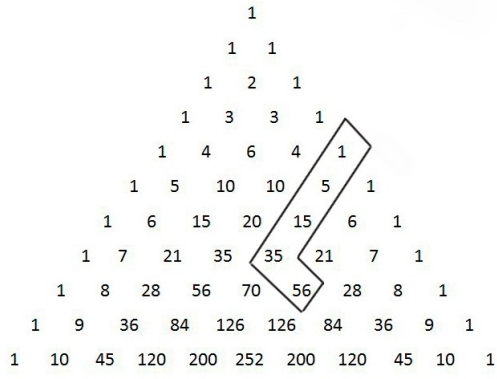
Fun with Pascal's triangle



Some simple observations. Recall: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?

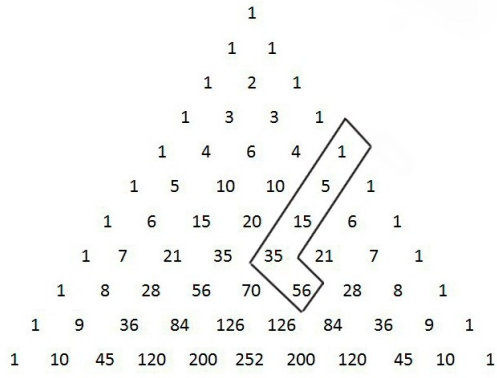
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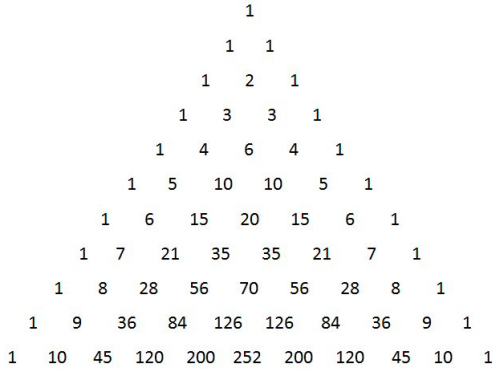
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3. Hockey stick patterns: (H.W-2) $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} \ldots + \binom{n-m}{0}$

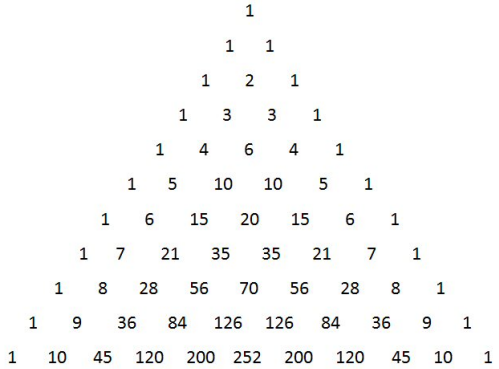
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Some not so simple observations

- For some rows, all values in row (except first and last) are divisible by second!

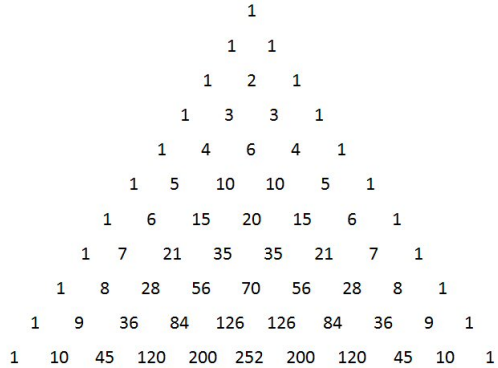
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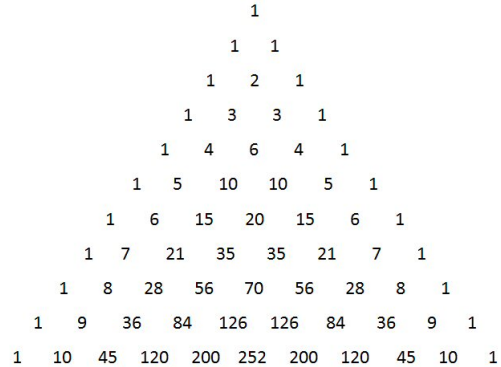
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- Corollary: $2^p - 2$ is a multiple of p, for any prime p.

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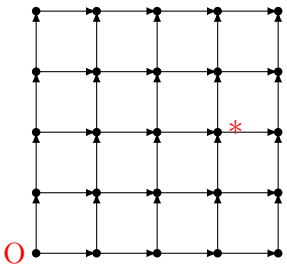
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- Interesting Ex.: Count no. of odd numbers in each row...

An application to path counting

Map problems

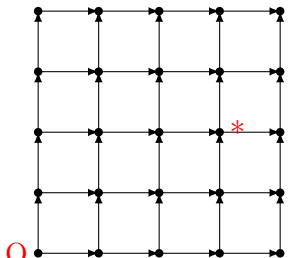
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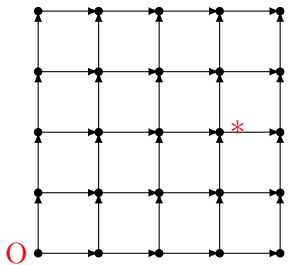


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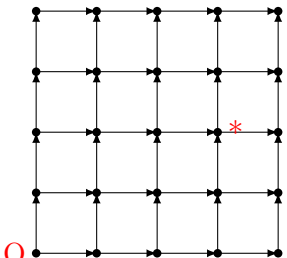


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- Our path has 5 segments of which there must be 3 R's.

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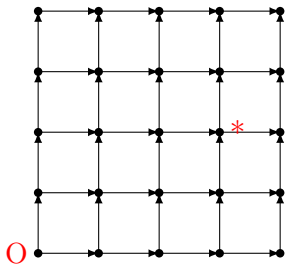


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- Bijection from set of paths to subsets of fixed size.
- No. of paths = ways of choosing 3 R's out of 5 elts = $\binom{5}{3}$.

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2. e.g., suppose we want to select 5 obj from a set of 4 with repetitions. Then, $**|*||**$ represents: 2 of first, 1 of second, none of third and 2 of fourth.

Combinations with repetitions

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- Ex: How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 17$ have such that $x_1, x_2, x_3, x_4 \in \mathbb{Z}^{\geq 0}$?

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How big is $n!$?

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Theorem (Stirling's Approximation)

$$e\left(\frac{n}{e}\right)^n \leq n! \leq ne\left(\frac{n}{e}\right)^n$$

where e is the base of natural logarithms, $\log(e) = e^{\log(e)} = 1$.