CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Sep 30, 2025

Lecture 15 – Counting and Combinatorics

Counting and Combinatorics

Topics to be covered

- Basics of counting
  - Product principle
  - Sum principle
  - Bijection principle
  - Double counting
- Subsets, partitions, Permutations and combinations
  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions
  4. Estimating $n!$

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  4. Estimating $n!$
- Today: Recurrence relations and generating functions

Solving general linear recurrence sequences

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- Recall the recurrence for Catalan Numbers: $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$, $C(0) = 0$, $C(1) = 1$.
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Reading assignment

Read examples/generalizations from Sections 6.1 and 6.2 from Rosen's book (7th Edition).

Proof Method 2: Using generating functions

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thus, as before

$$F(n) = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Properties of generating functions

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The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

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If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
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Class work/Pop Quiz!:

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Class work/Pop Quiz!:

1. Solve the recurrence $a_k = 4a_{k-1}$ with $a_0 = 3$.
2. **A coding problem:** A cryptographer builds a system where a string of decimals is codeword if it contains an even number of 0s. E.g., 1023038 is valid but not 10244.

   Let a_n be the number of n-digit codewords. Empty string is also valid.

   2.1 Find a recurrence relation for a_n. What is a_0, a_1?

   2.2 Solve the recurrence using generating functions.

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  - Compare coefficients of x^n in $(1 + x)^{2n} = ((1 + x)^n)^2$.
- (H.W) How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!

Solving Catalan numbers using generating functions

Catalan Numbers

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- Now consider $\phi(x)^2$.
- $\phi(x)^2 = (\sum_{k=1}^{\infty} C(k)x^k)(\sum_{k=1}^{\infty} C(k)x^k)$
  $= (\sum_{k=2}^{\infty} \sum_{i=1}^{k-1} C(i)C(k-i)x^k)$
  $= (\sum_{k=2}^{\infty} C(k)x^k) = \phi(x) - x$

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Catalan Numbers

$$C(n) = \sum_{i=1}^{n-1} C(i)C(n-i) \text{ for } n > 1,\ C(0) = 0, C(1) = 1.$$

- Let $\phi(x) = \sum_{k=1}^{\infty} C(k)x^k$.
- Now consider $\phi(x)^2$.
- $\phi(x)^2 = (\sum_{k=1}^{\infty} C(k)x^k)(\sum_{k=1}^{\infty} C(k)x^k)$
  $= (\sum_{k=2}^{\infty} \sum_{i=1}^{k-1} C(i)C(k-i)x^k)$
  $= (\sum_{k=2}^{\infty} C(k)x^k) = \phi(x) - x$
- Solving for $\phi(x)$ we get, $\phi(x) = \frac{1}{2}(1 \pm (1-4x)^{1/2})$
- But since $\phi(0) = 0$, we have $\phi(x) = \frac{1}{2}(1 - (1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2})$.

Catalan numbers

Recall: Extended binomial theorem

Let $\alpha \in \mathbb{R}, (1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$.

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- $\sum_{k=1}^{\infty} C(k)x^k = \phi(x) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}\sum_{k=0}^{\infty} \binom{1/2}{k}(-4x)^k)$.

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- The coefficient of x^k is $C(k) = -\frac{1}{2}\binom{1/2}{k}(-4)^k$
  $= -\frac{1}{2}(\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-k+1))\frac{(-4)^k}{k!}$
  $= -\frac{1}{2}(\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-\frac{2k-3}{2}))\frac{(-4)^k}{k!}$

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- $C(k) = \frac{(-1)^k(-4)^k}{2^{k+1}k!} \cdot 1 \cdot 3 \cdots (2k-3)$

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- $C(k) = \frac{1 \cdot 4^k}{2^{k+1} \cdot k!} \cdot \frac{1.2....(2k-3)(2k-2)}{2^{k-1}(k-1)!} = \frac{(2k-2)!}{k!(k-1)!}$.

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Thus, the n^{th} Catalan number is given by

$C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$