CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Oct 07, 2025

Lecture 17 – Counting and Combinatorics

Logistics

Quiz 2: Oct 23 Thursday 3.45pm

- Syllabus: Counting and Combinatorics, bit of Graph theory
- Venue: LH 301, LH 302, LT 305

Logistics

Quiz 2: Oct 23 Thursday 3.45pm

- Syllabus: Counting and Combinatorics, bit of Graph theory
- Venue: LH 301, LH 302, LT 305

Pop Quiz: Anytime

Applications of PHP

The coloring game

- There are six points on board and two colored chalks.

Applications of PHP

The coloring game

- There are six points on board and two colored chalks.
- Divide class into 2 groups. Group 1 draws white lines and Group 2 draws blue lines between points.

Applications of PHP

The coloring game

- There are six points on board and two colored chalks.
- Divide class into 2 groups. Group 1 draws white lines and Group 2 draws blue lines between points.
- You lose if you are first to draw a triangle of your color.

Applications of PHP

The coloring game

- There are six points on board and two colored chalks.
- Divide class into 2 groups. Group 1 draws white lines and Group 2 draws blue lines between points.
- You lose if you are first to draw a triangle of your color.
- Can you ever have a draw?

The coloring game

We will now show that this is impossible. That is,

The coloring game

We will now show that this is impossible. That is,

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

- 2-coloring of edges: coloring all edges of the graph using atmost 2 colors.
- monochromatic (triangle): all edges have the same color.

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle. □

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle. □
- What if there were 5 or lesser nodes?

The coloring game

Lemma

Any 2-coloring of edges of a graph on 6 nodes has a monochromatic triangle.

Proof:

- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.
- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.
- Now there are two possibilities:
  - Either one of $12, 23, 31$ is red (then we have a red triangle).
  - Else none of them are red, implies 123 is a blue triangle. □
- What if there were 5 or lesser nodes?

Optimality: 6 is the smallest such number

For any graph on 5 or less nodes the above lemma does not hold.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

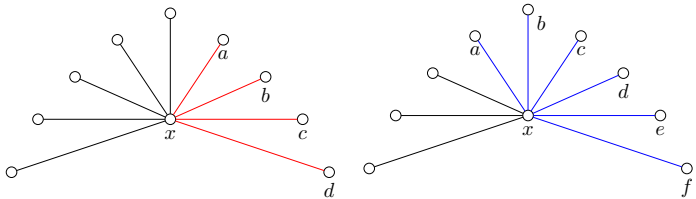
- complete: all pairs of edges are present.
- How do you prove this? Any ideas?
- How is this different from the previous problem?

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:



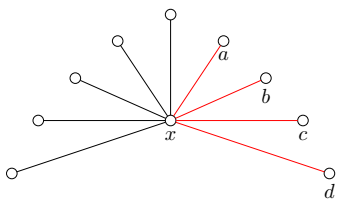
- Consider all edges from some node x.
- Either ≥ 4 edges have red color or ≥ 6 have blue (why?).

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:



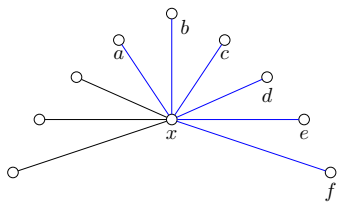
- Case 1: ≥ 4 red edges
  - Either one of edges between a, b, c, d is red or all are blue. So, we are done.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:



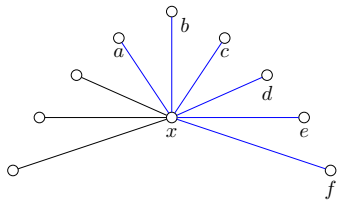
- Case 2: ≥ 6 blue edges
  - But this means that there are 6 nodes $a, \ldots f$.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:



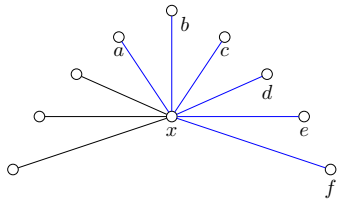
- Case 2: ≥ 6 blue edges
  - But this means that there are 6 nodes $a, \ldots f$.
  - Any 2-coloring on 6 vertices has a red or blue triangle.
  - Thus we are done again.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:



- Case 2: ≥ 6 blue edges
  - But this means that there are 6 nodes $a, \ldots f$.
  - Any 2-coloring on 6 vertices has a red or blue triangle.
  - Thus we are done again.
- And this completes the proof. □

Another coloring problem...

Thus, we have showed...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Another coloring problem...

Thus, we have showed...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- But, is this optimal?

Another coloring problem...

Thus, we have showed...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

Another coloring problem...

Thus, we have showed...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

Answer: No! In fact, it does hold on 9 nodes!

Another coloring problem...

Thus, we have showed...

Theorem
Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

Answer: No! In fact, it does hold on 9 nodes!

Theorem
Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Yet another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:

- Try the same proof as above?

Yet another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:

- Try the same proof as above?
- The only new case, where the previous proof does not work, is if all nodes have 3 red edges and 5 blue edges. (Why?)

Yet another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:

- Try the same proof as above?
- The only new case, where the previous proof does not work, is if all nodes have 3 red edges and 5 blue edges. (Why?)
- But is this case possible?

Yet another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:

- Try the same proof as above?
- The only new case, where the previous proof does not work, is if all nodes have 3 red edges and 5 blue edges. (Why?)
- But is this case possible?
- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.

Yet another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.

Proof:

- Try the same proof as above?
- The only new case, where the previous proof does not work, is if all nodes have 3 red edges and 5 blue edges. (Why?)
- But is this case possible?
- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.
- Thus, this case is impossible and we are done. □

Edge coloring problems

Summary of results till now

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number.

Edge coloring problems

Summary of results till now

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number.
2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

Edge coloring problems

Summary of results till now

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number.
2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.
3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number?
   - (H.W?) Prove that it is!

Edge coloring problems

Summary of results till now

1. Any 2-coloring of a graph on 6 nodes has either a red triangle or a blue triangle.
   - 6 is the optimal such number.
2. Any 2-coloring of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.
3. Any 2-coloring of a graph on 9 nodes has either a red triangle or a blue complete graph on 4 nodes.
   - Is 9 the optimal such number?
   - (H.W?) Prove that it is!

- (H.W) Prove that any 2-coloring of a graph on 18 nodes has a monochromatic complete graph on 4 nodes.
  (hint: you may use any of the above results)

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

- The minimal such number is denoted $R(k, \ell)$ and is called Ramsey number.

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

- The minimal such number is denoted $R(k, \ell)$ and is called Ramsey number.
- We have seen that $R(3, 3) = 6$.
- Also $R(3, 4) = 9$.

Can we generalize the above?

In general,

How many nodes should a (complete) graph have so that any 2 coloring of its edges has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

- The minimal such number is denoted $R(k, \ell)$ and is called Ramsey number.
- We have seen that $R(3, 3) = 6$.
- Also $R(3, 4) = 9$.

What about $R(k, \ell)$ in general?

Ramsey's theorem



Figure: Frank Plumpton Ramsey (1903-1930)

Ramsey's theorem



Figure: Frank Plumpton Ramsey (1903-1930)

Ramsey's theorem (simplified version)

Ramsey's theorem



Figure: Frank Plumpton Ramsey (1903-1930)

Ramsey's theorem (simplified version)

For any $k, \ell \in \mathbb{N}$, there exists $R(k, \ell) \in \mathbb{N}$ such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

Ramsey's theorem



Figure: Frank Plumpton Ramsey (1903-1930)

Ramsey's theorem (simplified version)

For any $k, \ell \in \mathbb{N}$, there exists $R(k, \ell) \in \mathbb{N}$ such that any 2-coloring of a (complete) graph on $R(k, \ell)$ nodes has

- either, a k-sized complete graph with all red edges
- or, a ℓ-sized complete graph with all blue edges

Moreover, we have

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

- Suppose in a group of people any two are friends or enemies.

Ramsey theory: A search for order in disorder!

Every structure no matter how disordered must contain some regular sub-part!

E.g., any 2-coloring on a complete graph of 10 nodes contains either a complete graph of 3 nodes of one color or a complete graph of 4 nodes of the other color.

- Suppose in a group of people any two are friends or enemies.
- In any set of 10 people there must be either 3 mutual friends or 4 mutual enemies.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof:

- By strong induction on $k + \ell$.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.

Proof of Ramsey's theorem

- What is $R(n, 2) = R(2, n)$?

For all integers $k, \ell \geq 2$, $R(k, \ell)$ is finite.

Proof:

- By strong induction on $k + \ell$.
- Base case: $R(2, 2) = 2$.
- Suppose it is true for all k, ℓ such that $k + \ell < N$. We will show that $R(k, \ell)$ is finite by showing

$$R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$$

Indeed, we can use induction to conclude that $R(k, \ell)$ is finite, since by induction hypothesis $R(k - 1, \ell)$ and $R(k, \ell - 1)$ exist (i.e., are finite) as $k + \ell - 1 < N$.

Proof of Ramsey's theorem contd.

Claim: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$

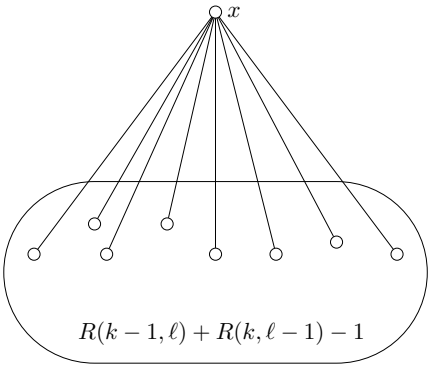
Proof of Ramsey's theorem contd.

Claim: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$

- i.e, we just need to show:
  Any 2-colored complete graph of $R(k-1, \ell) + R(k, \ell-1)$ nodes, has a complete red graph of k nodes or a complete blue graph of ℓ nodes.

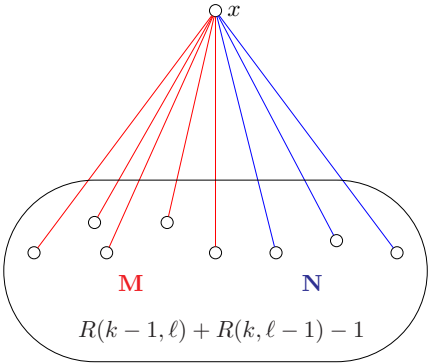
Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.



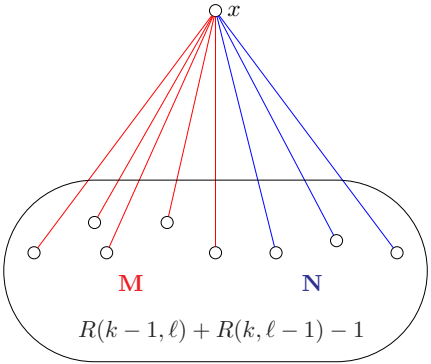
Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



Proof of Ramsey's theorem contd.

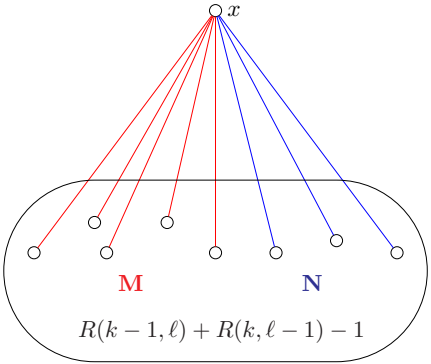
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Clearly $M+N=R(k-1,\ell)+R(k,\ell-1)-1$.

Proof of Ramsey's theorem contd.

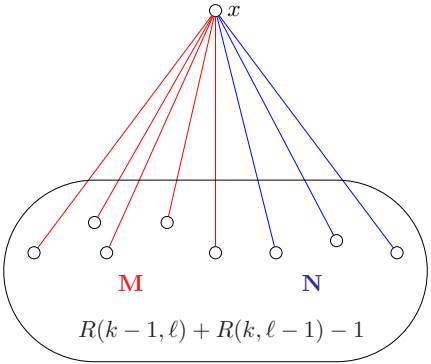
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Clearly $M+N=R(k-1,\ell)+R(k,\ell-1)-1$.
- Either $M \geq R(k-1,\ell)$ or $N \geq R(k,\ell-1)$ (PHP!)

Proof of Ramsey's theorem contd.

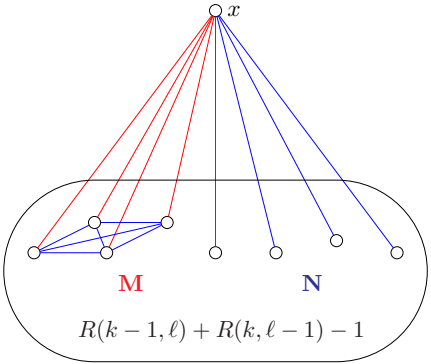
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Case 1: $M \geq R(k-1,\ell)$.

Proof of Ramsey's theorem contd.

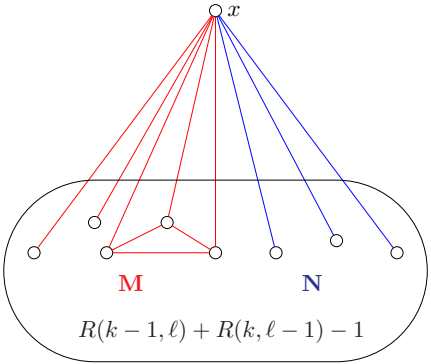
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Case 1: $M \geq R(k-1,\ell)$. Either complete blue graph on ℓ nodes

Proof of Ramsey's theorem contd.

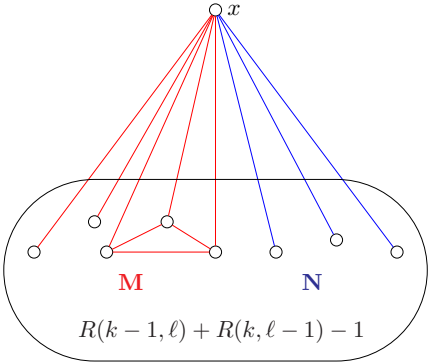
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Case 1: $M \geq R(k-1,\ell)$. Either complete blue graph on ℓ nodes or complete red graph on $k-1$ nodes + x

Proof of Ramsey's theorem contd.

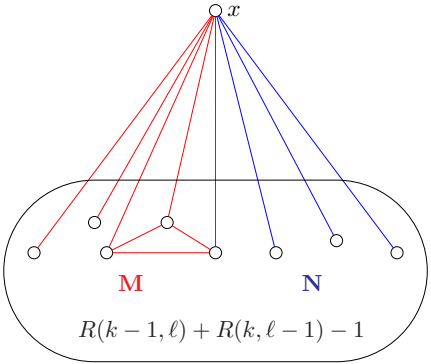
Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



- Case 1: $M \geq R(k-1,\ell)$. ✓
- Case 2: $N \geq R(k,\ell-1)$ leads to same argument.(Do it!) ✓

Proof of Ramsey's theorem contd.

Consider complete graph with $R(k-1,\ell)+R(k,\ell-1)$ nodes.
WTS It has k-node complete red or ℓ node complete blue graph



Thus in all cases, we have $R(k,\ell) \leq R(k-1,\ell)+R(k,\ell-1)$. □

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof: Now, this should be trivial!

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

- By induction on $k + \ell$ as before.

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

- By induction on $k + \ell$ as before.
- Base case for $k = \ell = 2$ is done.

Proof of Ramsey's theorem

Ramsey's theorem (simplified version)

For all $k, \ell \geq 2$, $R(k, \ell)$ exists, i.e., it is finite. Further,

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Proof:

- By induction on $k + \ell$ as before.
- Base case for $k = \ell = 2$ is done.
- By what we just showed and induction hypothesis we have:

$$\begin{aligned} R(k, \ell) &\leq R(k - 1, \ell) + R(k, \ell - 1) \\ &\leq \binom{k + \ell - 3}{k - 2} + \binom{k + \ell - 3}{k - 1} = \binom{k + \ell - 2}{k - 1} \end{aligned}$$

□

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,... $R(3,10) = 40$ or 41...
- Only bounds are known for rest. (see wiki on this...)

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,... $R(3,10) = 40$ or 41...
- Only bounds are known for rest. (see wiki on this...)
- What about lower bounds?

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,... $R(3,10) = 40$ or 41...
- Only bounds are known for rest. (see wiki on this...)
- What about lower bounds?

So how hard is it? Paul Erdös is supposed to have said:

Ramsey theory

Some interesting facts

- The general Ramsey theorem extends this to any finite number of colors (not just 2).
- Several applications, vast research area!
- Exact values are known only for 6 or so entries: $R(3,3) = 6$, $R(3,4) = 9$, $R(4,4) = 18$,... $R(3,10) = 40$ or 41...
- Only bounds are known for rest. (see wiki on this...)
- What about lower bounds?

So how hard is it? Paul Erdös is supposed to have said:

> *Suppose an evil alien would tell mankind "Either you tell me the value of $R(5,5)$ or I will exterminate the human race." ... It would be best to try to compute it, both by mathematics and with a computer. If he would ask for the value of $R(6,6)$, the best thing would be to destroy him before he destroys us, because we couldn't.*