CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Oct 10, 2025

Lecture 18 – An Introduction to Graph Theory

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs,

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs, regular= monochromatic/complete

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs, regular= monochromatic/complete

Next topic

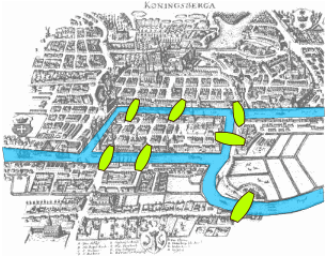
Graphs and their properties!

Topic 3: Graph theory

Textbook Reference

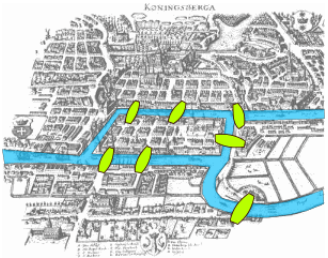
- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

Königsberg Bridge problem



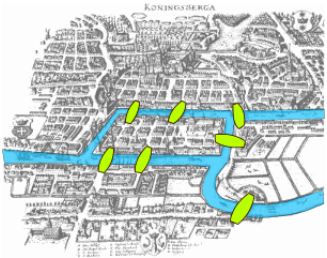
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.

Königsberg Bridge problem



- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- They couldn't find the answer, so in 1735 they asked Leonard Euler, mathematician in St. Petersburg.

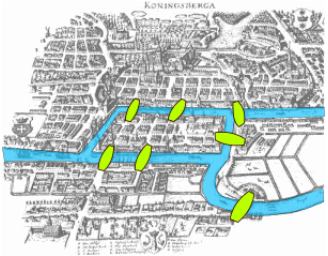
Königsberg Bridge problem





- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- They couldn't find the answer, so in 1735 they asked Leonard Euler, mathematician in St. Petersburg.

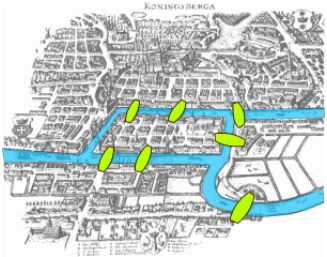
Königsberg Bridge problem





- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- "*This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.*"

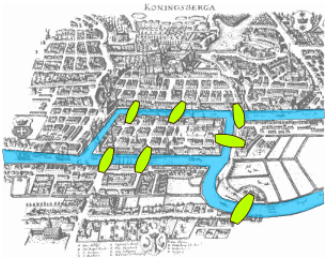
Königsberg Bridge problem





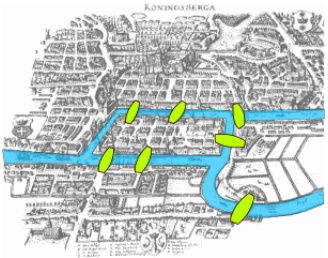
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- *"This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it."*
- Still, he wrote a paper showing that this is impossible!
- Thus, he "gave birth" to the area of graph theory.

Königsberg Bridge problem



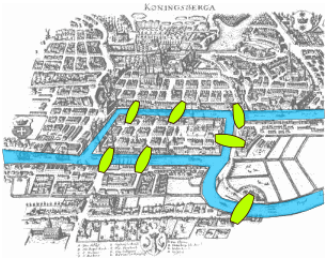
- Qn: Is there a walk that crosses every bridge exactly once and returns home?

Königsberg Bridge problem



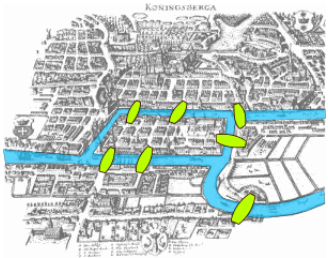
- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible

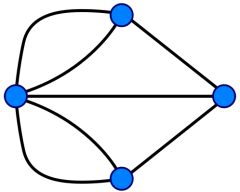
Königsberg Bridge problem



- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a “graph” (only what matters!)

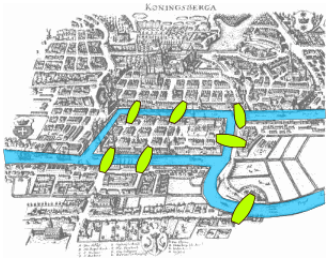
Königsberg Bridge problem

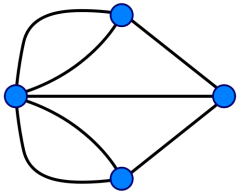




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a “graph” (only what matters!)

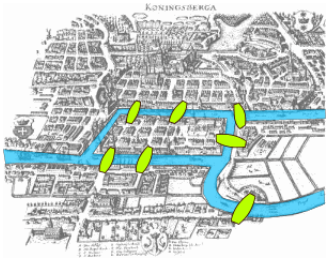
Königsberg Bridge problem

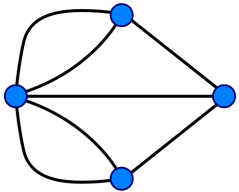




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.

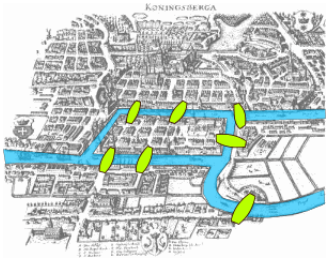
Königsberg Bridge problem

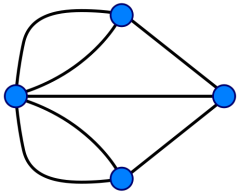




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.

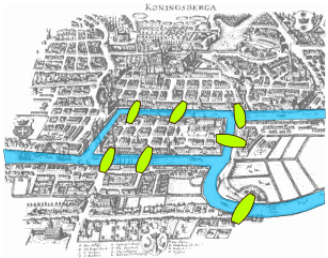
Königsberg Bridge problem

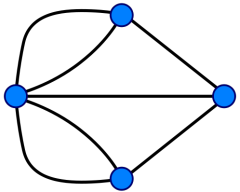




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible. □

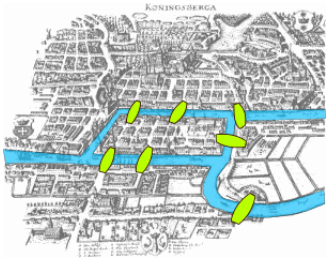
Königsberg Bridge problem

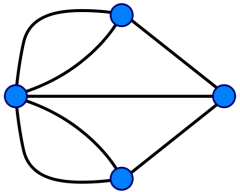




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible. □
- This is a sufficient condition, but is it necessary?

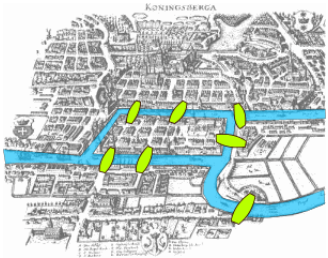
Königsberg Bridge problem

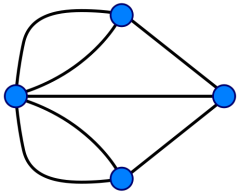




- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible. □
- This is a sufficient condition, but is it necessary?
- If every vertex is connected to an even no. of vertices in a graph, is there such a walk?

Königsberg Bridge problem





- Qn: Is there a walk that crosses every bridge exactly once and returns home?
- Leonard Euler showed in 1735 that this is impossible
  - First step: Reformulate into a "graph" (only what matters!)
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible. □
- This is a sufficient condition, but is it necessary?
- If every vertex is connected to an even no. of vertices in a graph, is there such a walk? This is called Eulerian walk.

What are graphs

What are graphs

Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

What are graphs

Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

- What about loops and directed/multiple edges?

What are graphs

Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

- What about loops and directed/multiple edges?

General Definition

A graph G is a triple (V, E, R) where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

What are graphs

Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

- What about loops and directed/multiple edges?

General Definition

A graph G is a triple (V, E, R) where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often simple graphs.

What are graphs

Definition

A simple graph G is a pair (V, E), where

- V is a set of vertices/nodes, and,
- E is a set of edges between unordered pairs of vertices (i.e., end-points): $e = vu$ means e is edge between v, u ($u \neq v$)

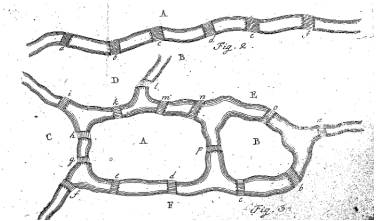
- What about loops and directed/multiple edges?

General Definition

A graph G is a triple (V, E, R) where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

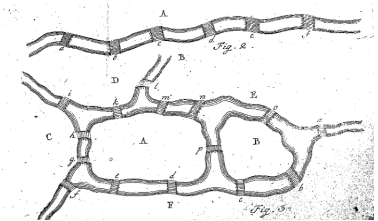
We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often simple graphs. Also, we assume $|V| > 0$.

Basic terminology



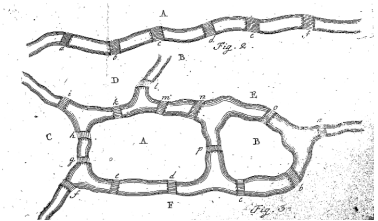
Exercise. Draw the graphs!

Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

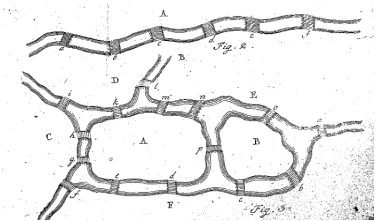
Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \dots v_k$ such that $\forall i \in \{1, \dots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.

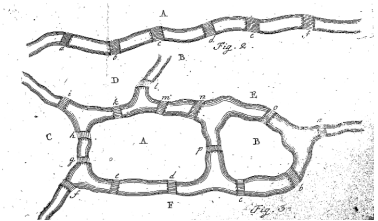
Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \dots v_k$ such that $\forall i \in \{1, \dots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.
- A walk is called closed if it starts and ends with the same vertex.

Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \dots v_k$ such that $\forall i \in \{1, \dots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.
- A walk is called closed if it starts and ends with the same vertex.
- A Graph is connected if there is a walk between any two vertices of the graph.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof:

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof: ($\Longrightarrow$)

- Suppose G is Eulerian: every vertex has even degree.
  - each passage through a vertex uses two edges (in and out).
  - at the first vertex first edge is paired with last.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof: ($\Longrightarrow$)

- Suppose G is Eulerian: every vertex has even degree.
  - each passage through a vertex uses two edges (in and out).
  - at the first vertex first edge is paired with last.
- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).

A quick exercise

A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

A quick exercise

A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

A quick exercise

A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:



- Let P be a maximal path in G. It starts from some u.

A quick exercise

A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Prove or disprove

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:



- Let P be a maximal path in G. It starts from some u.
- Since P cannot be extended, every nbr of u is already in P.

A quick exercise

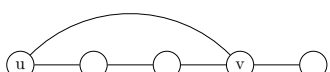
A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:



- Let P be a maximal path in G. It starts from some u.
- Since P cannot be extended, every nbr of u is already in P.
- As $d(u) \geq 2, \exists v$ in P such that $uv \notin P$. Thus, we have cycle $u \dots vu$. □

A quick exercise

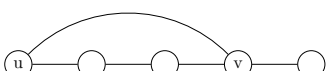
A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:



- Let P be a maximal path in G. It starts from some u.
- Since P cannot be extended, every nbr of u is already in P.
- As $d(u) \geq 2, \exists v$ in P such that $uv \notin P$. Thus, we have cycle $u \dots vu$. □

Is this true if G were infinite?

A quick exercise

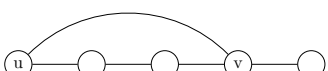
A few more definitions

- A path is a walk in which no vertex is repeated. A path P is maximal in G if it is not contained in any longer path. In a finite graph maximal paths exist.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:



- Let P be a maximal path in G. It starts from some u.
- Since P cannot be extended, every nbr of u is already in P.
- As $d(u) \geq 2, \exists v$ in P such that $uv \notin P$. Thus, we have cycle $u \dots vu$. □

Is this true if G were infinite? No! Consider $V = \mathbb{Z}$, $E = \{ij : |i - j| = 1\}$.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Base case: $m = 1$.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.
  - Get $G_1, \ldots, G_k$. Each G_i is
    - connected
    - has $< m$ edges.
    - all its vertices have even degree (why? – degree of any vertex was even and removing C, reduces each vertex degree by 0 or 2.)

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.
  - Get $G_1, \ldots, G_k$. Each G_i is
    - connected
    - has $< m$ edges.
    - all its vertices have even degree (why? – degree of any vertex was even and removing C, reduces each vertex degree by 0 or 2.)

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.
  - By indn hyp, each G_i is Eulerian, has a Eulerian walk.

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.
  - By indn hyp, each G_i is Eulerian, has a Eulerian walk.
  - Combine the cycle C and the Eulerian walk of $G_1, \ldots, G_k$ to produce an Eulerian walk of G (how?).

Proof of theorem on Eulerian graphs

Lemma

If every vertex of a graph G has degree at least 2, then G contains a cycle.

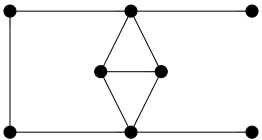
Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof ($\Longleftarrow$): By induction on number of edges m in G.

- Induction step: $m > 1$. By Lemma G has a cycle C.
  - Delete all edges in cycle C and (new) isolated vertices.
  - By indn hyp, each G_i is Eulerian, has a Eulerian walk.
  - Combine the cycle C and the Eulerian walk of $G_1, \ldots, G_k$ to produce an Eulerian walk of G (how?).
    - Traverse along cycle C in G and when some G_i is entered for first time, detour along an Eulerian walk of G_i.
    - This walk ends at vertex where we started detour.
    - When we complete traversal of C in this way, we have completed an Eulerian walk on G. □

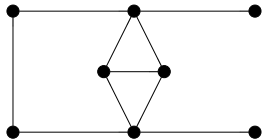
An Eulerian Puzzle!



Question

1. Can the above graph be drawn without lifting your pen from paper? No segment should be drawn twice.

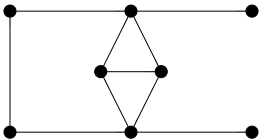
An Eulerian Puzzle!



Question

1. Can the above graph be drawn without lifting your pen from paper? No segment should be drawn twice.
2. How many times must you lift your pen to draw the graph?

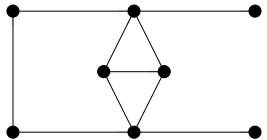
An Eulerian Puzzle!



Question

1. Can the above graph be drawn without lifting your pen from paper? No segment should be drawn twice.
2. How many times must you lift your pen to draw the graph?
3. For any connected graph G, what is the number of times we need to lift our pen to draw it?

An Eulerian Puzzle!



Question

1. Can the above graph be drawn without lifting your pen from paper? No segment should be drawn twice.
2. How many times must you lift your pen to draw the graph?
3. For any connected graph G, what is the number of times we need to lift our pen to draw it?

Hint1: What is the connection to Eulerian graphs?
Hint2: What happens if all nodes in the graph have even degree?