CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Oct 14, 2025

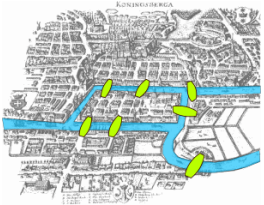
Lecture 19 – An Introduction to Graph Theory

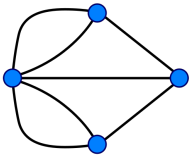
Topic 3: Graph theory

Textbook Reference

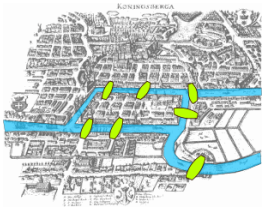
- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

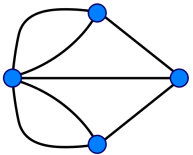
Defining graphs





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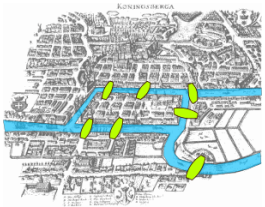


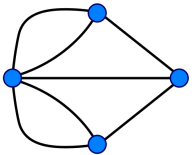


Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means e is an edge between v and u ($u \neq v$).

Defining graphs





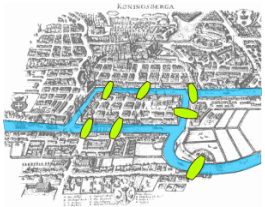
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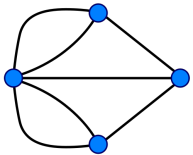
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General Definition

A graph G is a triple V, E, R where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation.

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We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often simple graphs. **Also, we assume** $|V| > 0$.

Basic terminology

- The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

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- A walk is a sequence of vertices $v_1, \ldots v_k$ such that $\forall i \in \{1, \ldots k-1\}$, $(v_i, v_{i+1}) \in E$. **Vertices v_1 and v_k are called end-points** and others are called internal vertices.

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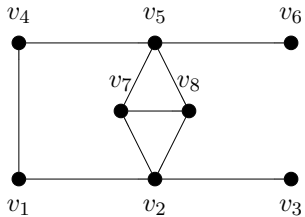
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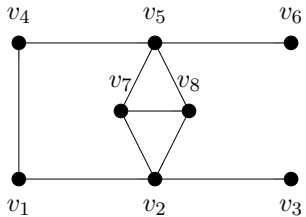
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- A graph is connected if there is a walk between every two vertices.
- **The length of a walk is the number of edges in it.**

Basic terminology: Post Lunch Quiz Part 1



1. Give examples for each of the following in the above graph:
   - 1.1 All vertices of degree 3
   - 1.2 A walk of length 5
   - 1.3 A closed walk of length 6

Basic terminology: Post Lunch Quiz Part 1



1. Give examples for each of the following in the above graph:
   - 1.1 All vertices of degree 3
   - 1.2 A walk of length 5
   - 1.3 A closed walk of length 6
2. True or False: (a) If a graph is not connected, it must have an isolated vertex. (b) A graph is connected iff some vertex has an edge to every other vertex.

Post Lunch Quiz: Part 2

More definitions

- A path is a walk in which no vertex is repeated.

Post Lunch Quiz: Part 2

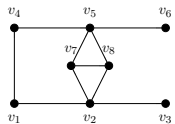
More definitions

- A path is a walk in which no vertex is repeated.
- A cycle is a (non-empty) closed walk whose internal vertices are all distinct from each other and from the end-point **and edges are also distinct**.
- A path P is said to be maximal in G if it is not contained in any longer path.

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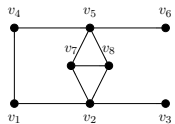


3. Give examples for each of the following in the above graph:
   3.1 A path of length 5 and a cycle of length 6.
   3.2 A maximal path that is not maximum.
4. *Every closed walk whose length is odd contains an cycle whose length is odd.

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More definitions

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3. Give examples for each of the following in the above graph:
   3.1 A path of length 5 and a cycle of length 6.
   3.2 A maximal path that is not maximum.
4. *Every closed walk whose length is odd contains an cycle whose length is odd. (contains means vertices and edges of path occur as a sublist of walk, in cyclic order but not necessarily consecutive.)

Some simple types of Graphs

- We have already seen some: connected graphs, complete graphs.

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Some simple types of Graphs

- We have already seen some: connected graphs, complete graphs.
- paths, cycles.
- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
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Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Exercises

1. What is largest n s.t n-vertex complete graph is bipartite?
2. Can a n-vertex bipartite graph have $n - 1$ degree vertex?
3. A graph is bipartite iff we can color vertices with 2 colors s.t no 2 adjacent vertices have same color.

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Exercise: Prove or Disprove:

Every closed odd walk contains an odd cycle.

Characterizing bipartite graphs using cycles.

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Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk.

Characterizing bipartite graphs using cycles.

Lemma

Every closed odd walk contains an odd cycle.

Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.

Proof:

- ($\implies$) direction is easy.

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- ($\implies$) direction is easy.
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$\implies$ if we start from X, each return to X can only happen after an even number of steps.

$\implies$ G has no odd cycles.

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- Let $u \in V$. Break V into

  $X = \{v \in V \mid \text{length of shortest path } P_{uv} \text{ from } u \text{ to } v \text{ is even}\}$,
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- If there is an edge vv' between two vertices of X or two vertices of Y, this creates a closed odd walk: $uP_{uv}vv'P_{v'u}u$.
- By Lemma, it must contain an odd cycle: contradiction.
- This along with $X \cap Y = \emptyset$ and $X \cup Y = V$, implies X, Y is a bipartition. $\square$