CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Graph theory

Matchings!

Lecture 21
Oct 24 2025

Logistics

Make up Lecture

- Date and Time: Monday, Oct 27 at 3.30pm
- Venue: LH 301

Topic 4: Graph theory (Recap)

1. Basics: graphs, paths, cycles, walks, trails.
2. Eulerian graphs: characterization using degrees of vertices.
3. Bipartite graphs: characterization using odd length cycles.

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1. Consider an "interesting" problem.
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A General Recipe

1. Consider an "interesting" problem.
2. Model it as a "special" class of graphs.
3. Characterize them using properties on vertices, cycles etc.
4. Not done in this course: Build algorithms based on the characterization, analyze and implement them!

This lecture, we will consider another new problem on graphs:

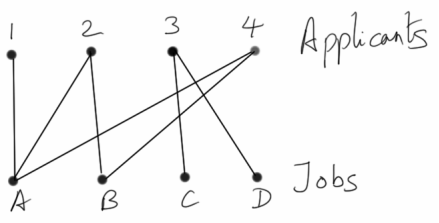
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- Suppose m people are applying for n different jobs. But not all applicants are qualified for all jobs, and each can hold at most one job.
- Then can you find a unique way to match jobs to applicants?

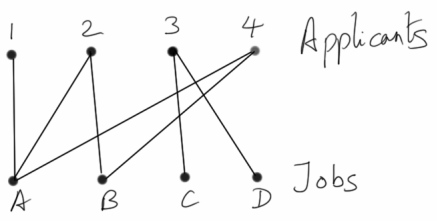
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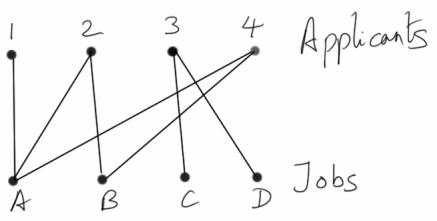
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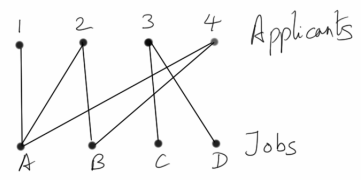
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- What are the properties of such an assignment?
- Another practical example: the dating scene!

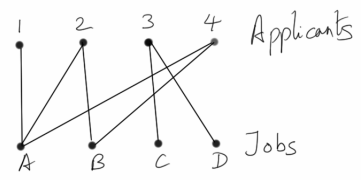
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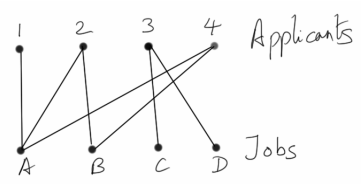
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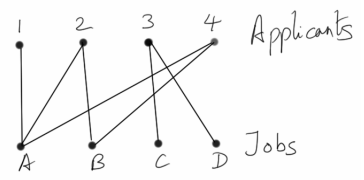
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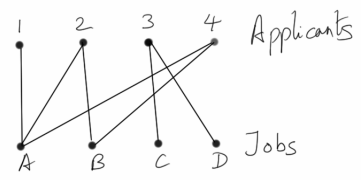


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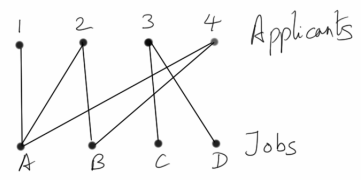


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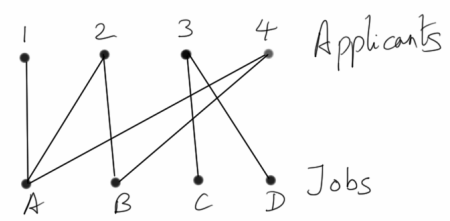


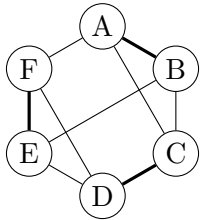
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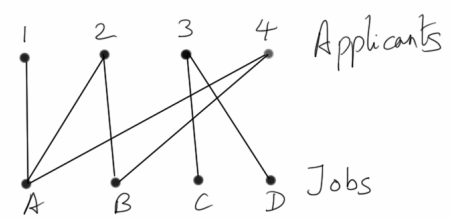
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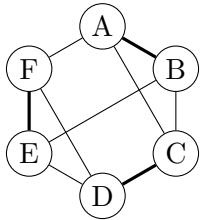




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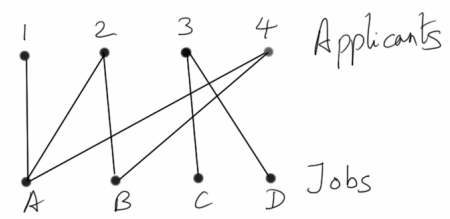
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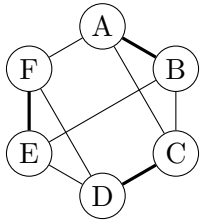




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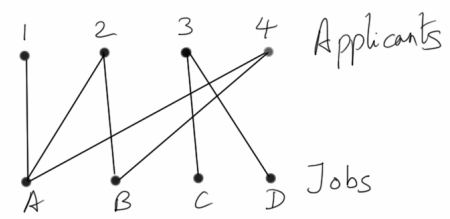
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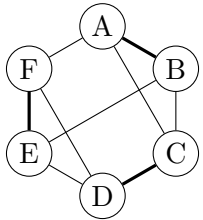




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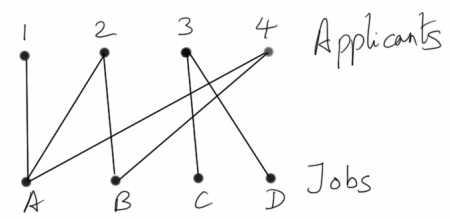
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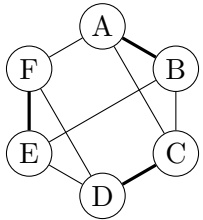




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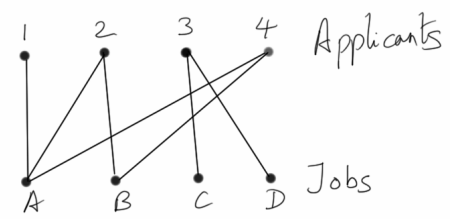


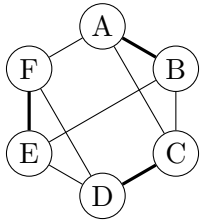


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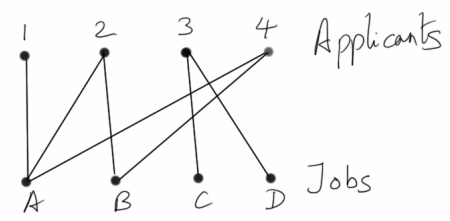


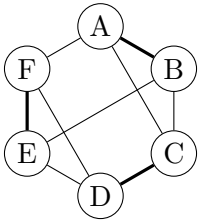
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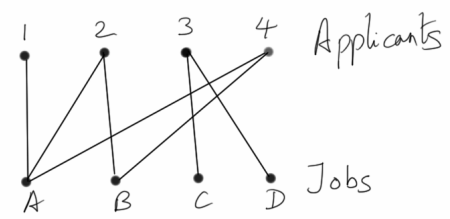


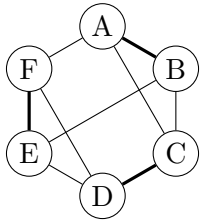
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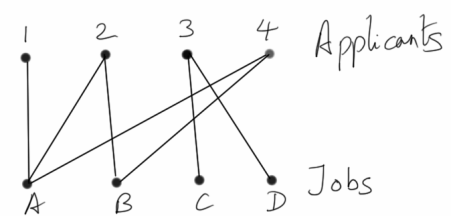


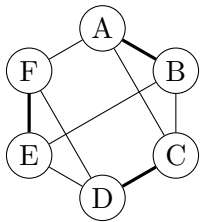
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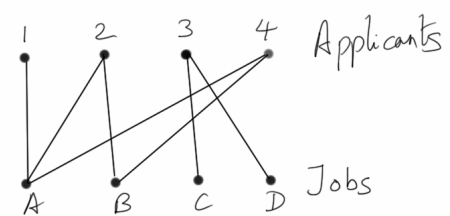
Maximum matchings

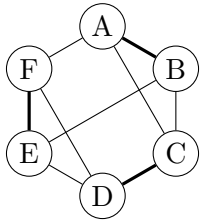




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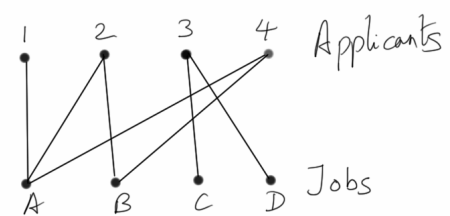


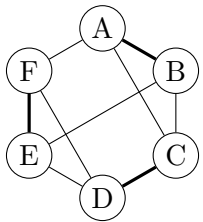
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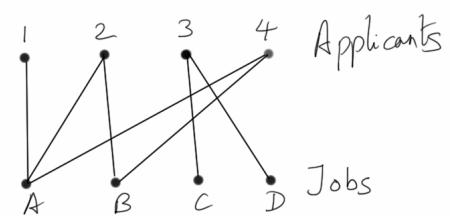


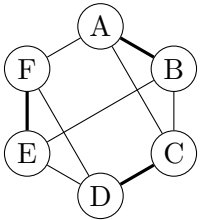
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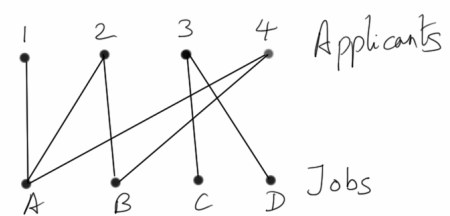


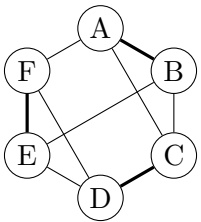
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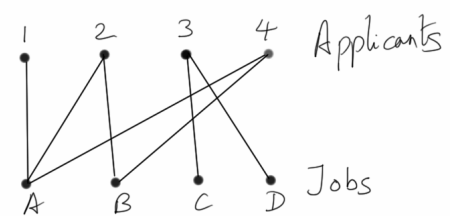


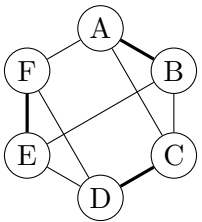
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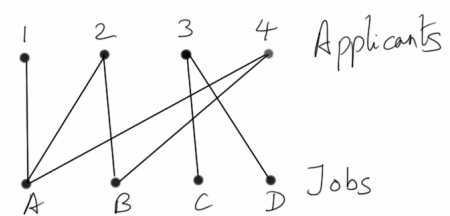
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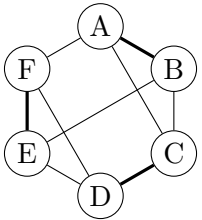
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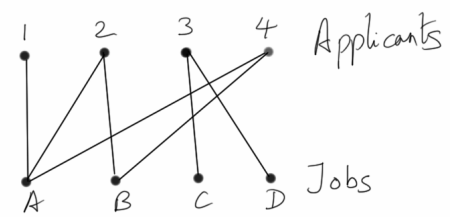
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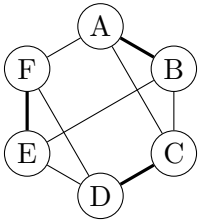
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- A maximum matching is a matching of maximum size (in terms of number of edges) among all matchings in a graph.

- A matching is maximal if every edge not in it is incident to some edge in it.
- Does maximum $\implies$ maximal? Does maximal $\implies$ maximum?

Maximum matchings





Perfect matchings are not always possible; so what is the best we can do?

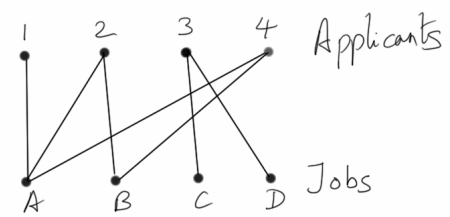
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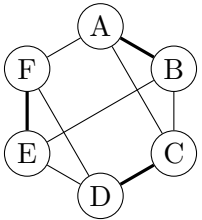
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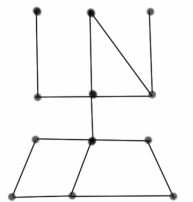
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Alternating and Augmenting paths

When do we know there must be a larger matching?

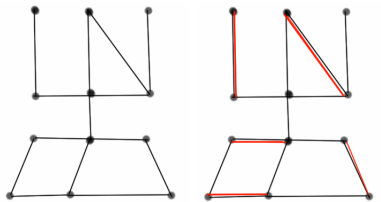
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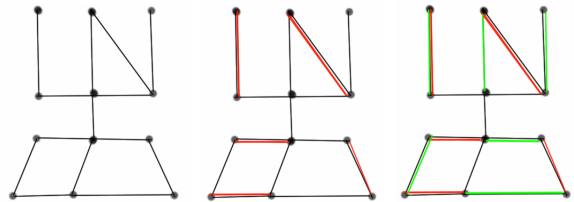
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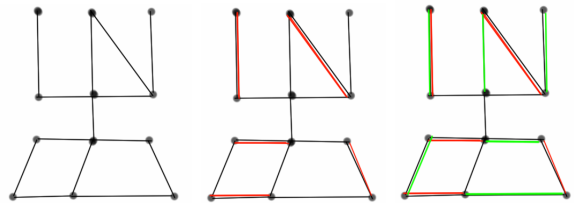
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Given a matching M,

- an M-alternating path is a path that alternates between edges in M and edges not in M.
- An M-alternating path whose endpoints are unmatched by M is an M-augmenting path.

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If a matching has an M-augmenting path, then can it be a maximum matching?

Characterization of Maximum matchings

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Theorem

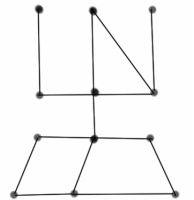
A matching M in G is a maximum matching iff G has no M-augmenting path.

Characterizing maximum matchings

We need a definition and a lemma.

Definition

If M, M' are matchings in a graph H, the symmetric difference $M \triangle M'$ is the set of edges which are either in M or in M' but not both, i.e.,
$M \triangle M' = (M \setminus M') \cup (M' \setminus M)$.

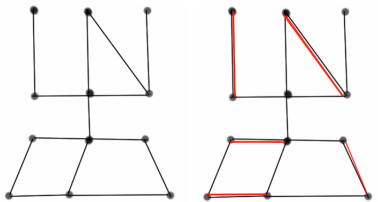


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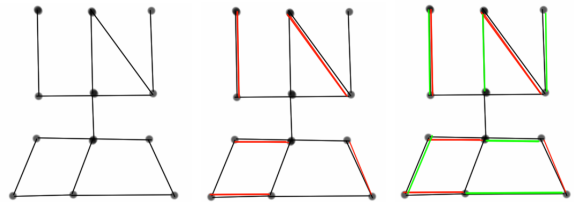


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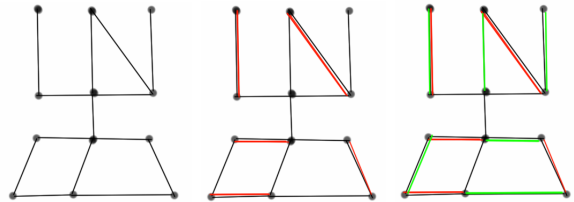
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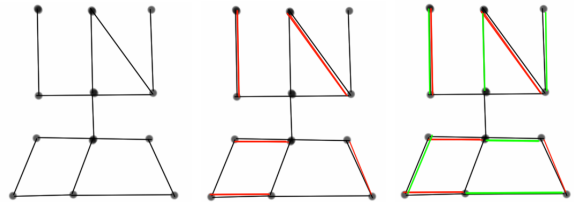
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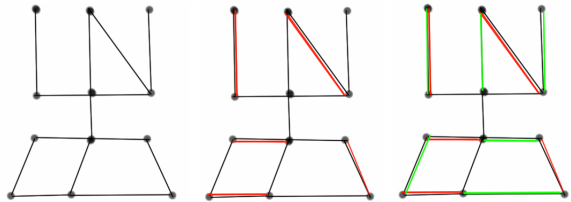


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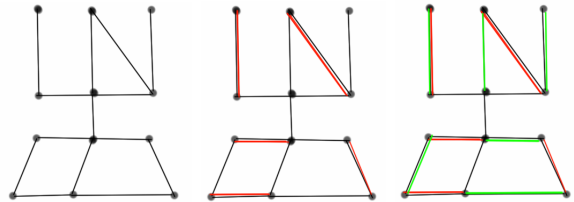


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- Further every path/cycle alternates between edges of $M \setminus M'$ and $M' \setminus M$.
- Thus, each cycle has even length with equal edges from M and M'. $\square$

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Theorem (Berge'57)

A matching M in G is a maximum matching iff G has no M-augmenting path.

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- But then since $|M'| > |M|$ it must have a component with more edges in M' than M.
- This component can only be a path that starts and ends with an edge of M'; i.e., it is an M-augmenting path in G.