CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Graph theory

Perfect Matchings and Hall's theorem

Lecture 22
Oct 27 2025

Topic 4: Graph theory

Basic definitions and concepts

- Basics: graphs, paths, cycles, walks, trails, ...
- Cliques and independent sets.
- Graph representations, isomorphisms and automorphisms.
- Matchings: perfect, maximal and maximum.

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5. Today: Perfect matchings in bipartite graphs

Recap: Matchings

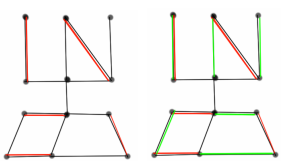
Definitions

- Matching: set of edges with no shared end-points.
- Vertices incident to edges in matching are saturated. Others are unsaturated.
- Perfect matching: saturates every vertex in graph.
- Maximum matching: matching of maximum size (# edges).
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Theorem

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Recap: Characterizing maximum matchings

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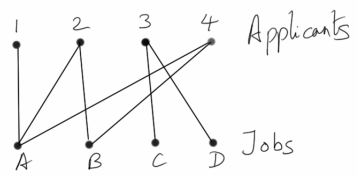
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- This component can only be a path that starts and ends with an edge of M'; i.e., it is an M-augmenting path in G. $\square$

Perfect matchings in bipartite graphs

- Pop-Quiz!: If there are n women and n men, and each woman is compatible with exactly k men and each man compatible with exactly k women, can they be perfectly matched?

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- If there are m jobs and n applicants, when can we find a perfect matching where all m jobs are saturated?



Perfect matchings in bipartite graphs

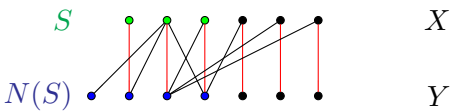
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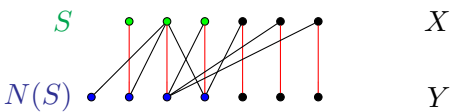
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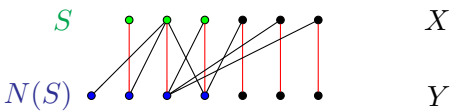
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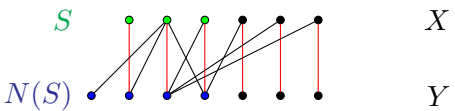
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- For $v \in V$, its neighbour-set $N(v) = \{u \in V \mid (u, v) \in E\}$.
- For $S \subseteq V$, $N(S) = \{u \in V \mid (u, v) \in E \text{ for some } v \in S\}$.

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Proof: ($\implies$) is straightforward:

- Let M be a matching.
- Then for any $S \subseteq X$, each vertex of S is matched to a distinct vertex in $N(S)$
- So $|N(S)| \geq |S|$.

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- Let M be such a maximum matching. Then, we will construct $S \subseteq X$ s.t. $|N(S)| < |S|$.

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Proof: ($\Longleftarrow$) Thus, starting from a maximum matching M which does not saturate X, we construct $S \subseteq X, |N(S)| < |S|$.

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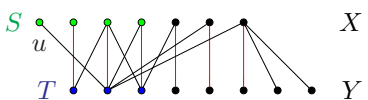
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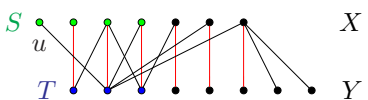
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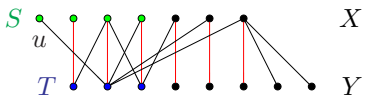
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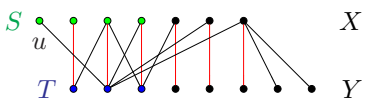
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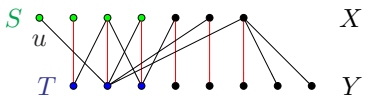
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- Thus, there is a bijection between T and $S \setminus \{u\}$.

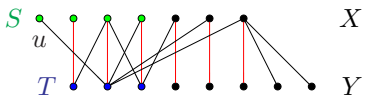
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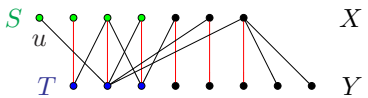
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- Conversely, if $v \in S$ has edge to $y \in Y \setminus T$, then path from u to v via M to y is an M-alternating path, implies $y \in T$.

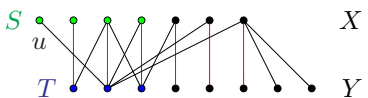
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Thus, $|N(S)| = |T| = |S| - 1 < |S|$ $\square$.

Applications of Hall's condition

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A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

The Marriage Theorem (1917)

- In a group of n women and n men, if every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist!
- What is the formal statement?

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- For $k > 0$, every k-regular bipartite graph (i.e, every vertex has degree exactly k) has a perfect matching. Ex. Prove this!

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- If G is a k-regular X, Y bipartite graph, then $|X| = |Y|$. (why?)

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- Can you now verify Hall's condition?

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  - Since G is k-regular, $m = k|S|$. And since they touch $N(S)$, $m \leq k|N(S)|$.

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- If a matching saturates X then it saturates Y.
  - Let $S \subseteq X$. Let $m = \#$ edges from S to $N(S)$.
  - Since G is k-regular, $m = k|S|$. And since they touch $N(S)$, $m \leq k|N(S)|$. Hence $k|S| \leq k|N(S)|, k > 0$ completes proof. □

Let's play a game

A two player game on a graph

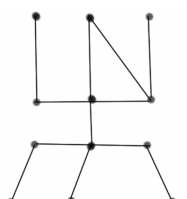
1. Given a graph G, two players will alternatively choose distinct vertices.
2. One player starts by choosing any vertex.
3. Subsequent move must be adjacent to preceding choice (of other player).
4. Last player who can move wins.

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Two volunteers please. Who wants to start?

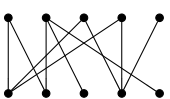


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Application 2: Theorem (H.W: Qn 3.1.18 from Douglas West)

If G has a perfect matching, then player 2 has a winning strategy; otherwise, player 1 has a winning strategy.