CS228 : Logic for Computer Science

Module 1: Propositional Logic

Lecture 4: Natural Deduction Rules

Instructor: S. Akshay

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Logistics

Quiz 1

- Aug 28th 5.10pm
- Venue: Most likely LH 101, 102, LT 105 (More details on Piazza)
- Syllabus: All material covered till Aug 25th.

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Problemsheets and help sessions

- First problemsheet is out!
- Tomorrow 3.30pm there is an optional help session at CC101, 103, 105.

Recap: Lectures 1 and 2

Lecture 1: Syntax

- Atomic propositions; logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus, \top, \bot$)
- Propositional formulas (defined inductively)

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- Propositional formulas (defined inductively)

Lecture 2: Parse Trees, Semantics & Problems of Interest

- Parse trees / DAGs for formulas
- Semantics via truth tables
- Satisfiability, validity, semantic entailment ($\models$)

Recap: Lecture 3

Towards Formal Proofs

- Semantic equivalence ($\equiv$); used it to simplify formulas
- Minimal/universal connectives
- Truth tables don't scale (2^{100} rows!)
- Need a formal proof system

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  - Sequents $F_1, \ldots, F_n \vdash G$ is valid if we can derive G from $F_1, \ldots, F_n$ using the rules of the proof system
    - Soundness: $F_1, \ldots, F_n \vdash G$ valid $\Rightarrow F_1, \ldots, F_n \models G$
    - Completeness: $F_1, \ldots, F_n \models G \Rightarrow F_1, \ldots, F_n \vdash G$ valid

Recap: Lecture 3

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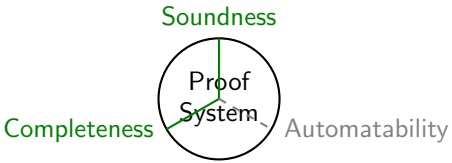
    - Completeness: $F_1, \ldots, F_n \models G \Rightarrow F_1, \ldots, F_n \vdash G$ valid
  - Started Natural deduction: saw the $\wedge$ rules



Topic 4.1

Natural Deduction Rules

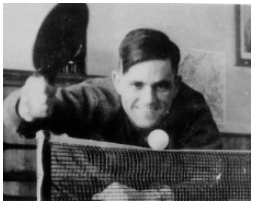
A formal proof system: Natural deduction

Natural deduction rules (from Huth & Ryan's Book)

12 Axioms or rules:

1. Introducing and Eliminating $\wedge$
2. Introducing and Eliminating $\rightarrow$
3. Introducing and Eliminating $\vee$
4. Negation rules: Introducing, Eliminating and Double negation
5. Bot rule

4 Derived/inferred rules to make life easier!



Gerhard Gentzen (1909–1945)

Conjunction Rules

If my cat is brown and it is fat then my cat is brown and fat!

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$$\frac{F \qquad G}{F \wedge G} \wedge i$$

$$\frac{F \wedge G}{F} \wedge e_1$$

$$\frac{F \wedge G}{G} \wedge e_2$$

- Rules to Introduce and Eliminate $\wedge$

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- **Exercise 4.1 Show that $p \wedge q, r \vdash q \wedge r$ is a valid sequent.**

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- **Exercise 4.1 Show that $p \wedge q, r \vdash q \wedge r$ is a valid sequent.**

1. $p \wedge q$ premise

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1.	$p \wedge q$	premise
2.	r	premise

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3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

When clear we omit subscripts e_1, e_2 and just write e.

Double negation rules

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Prove the sequent: $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ using only the rules till now!

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1. p premise

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2.	$\neg\neg(q \wedge r)$	premise

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4.	r	$\wedge e$ 3

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4.	r	$\wedge e$ 3
5.	$\neg\neg p$	$\neg\neg i$ 1

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6.	$\neg\neg p \wedge r$	$\wedge i$ 5,4

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4.	r	$\wedge e$ 3
5.	$\neg\neg p$	$\neg\neg i$ 1
6.	$\neg\neg p \wedge r$	$\wedge i$ 5,4

(Negation rules come later!)

Implication Rules

Whenever it rains, I carry an umbrella. It's raining — so I have my umbrella!

The implies elimination rule or Modus Ponens MP

$$\frac{F \qquad F \to G}{G} \to e \text{ or MP}$$

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3.	p	premise
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2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

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3.	p	premise
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5.	q	MP 2,3
6.		

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2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

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3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6

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Exercise 4.4

Show the following sequents are valid

1. $\neg p \to q, \neg q \vdash p$
2. $p \to \neg q, q \vdash \neg p$

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2.	q	premise

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2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2

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2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3

Implication Rules -3

- By MT, we have $p \rightarrow q, \neg q \vdash \neg p$.

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- By MT, we have $p \to q, \neg q \vdash \neg p$.
- This looks like the contrapositive! i.e., can we prove $p \to q \vdash \neg q \to \neg p$?

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- But we don't know how to introduce $\to$ yet!

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- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

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- But we don't know how to introduce $\to$ yet!
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, from premises AND assumption, using MT we get:

Implication Rules -3

- By MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- This looks like the contrapositive! i.e., can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- But we don't know how to introduce $\rightarrow$ yet!
- Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, from premises AND assumption, using MT we get:

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

Implication Rules -3 (Contd)

To convince someone that "if it rains, I'll get wet," you don't check every possible day — you just imagine it raining, and show that you'd get wet!

$$\frac{\boxed{\begin{array}{c} F \\ \vdots \\ G \end{array}}}{F \to G} \to i$$

- Can use F and any premises so far; Can nest boxes, have multiple boxes etc.
- Line immediately after a closed box must match the pattern!

Implication Rules -3 (Contd)

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- F and G can be same!

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1.	p	assumption
2.	$p \to p$	$\to i$ 1-1

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- We write $\vdash p \to p$ to denote it doesn't depend on any premises.

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2.	$p \to p$	$\to i$ 1-1

- We write $\vdash p \to p$ to denote it doesn't depend on any premises.
- Formula F such that the sequent $\vdash F$ is valid is called a theorem.

Implication Rules -3 (Contd)

In English: "If mammals breathe air, and non-mammals aren't whales, then whales breathe air!"
Example: Using all the rules, show that $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$

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1.	$q \to r$	assumption
2.		
	$(\neg q \to \neg p) \to (p \to r)$	
10.	$(q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$	$\to i$ 1-9

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1.	$q \to r$	assumption
2.	$\neg q \to \neg p$	assumption
3.		
	$p \to r$	
9.	$(\neg q \to \neg p) \to (p \to r)$	$\to i$ 2-8
10.	$(q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$	$\to i$ 1-9

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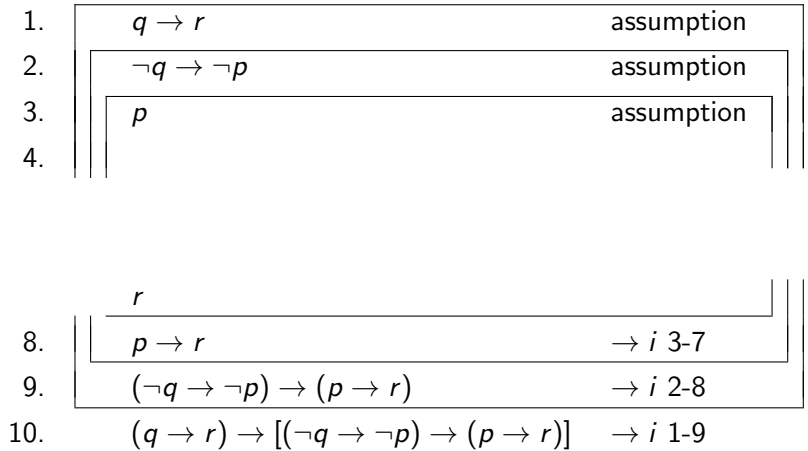

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1.	$q \to r$	assumption
2.	$\neg q \to \neg p$	assumption
3.	p	assumption
4.		
	q	
	r	MP 1,6
8.	$p \to r$	$\to i$ 3-7
9.	$(\neg q \to \neg p) \to (p \to r)$	$\to i$ 2-8
10.	$(q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$	$\to i$ 1-9

Implication Rules -3 (Contd)

In English: "If mammals breathe air, and non-mammals aren't whales, then whales breathe air!"

Example: Using all the rules, show that $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$

1.	$q \to r$	assumption
2.	$\neg q \to \neg p$	assumption
3.	p	assumption
4.	$\neg\neg p$	$\neg\neg i$ 3
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3.	p	assumption
4.	$\neg\neg p$	$\neg\neg i$ 3
5.	$\neg\neg q$	MT 2,4
6.	q	
	r	MP 1,6
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5.	$\neg\neg q$	MT 2,4
6.	q	$\neg\neg e$ 5
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Converts proof of $F_1, \ldots, F_n \vdash G$ to $\vdash F_1 \to (F_2 \to (\ldots (F_n \to G) \ldots))$

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1.	$(p \to q) \wedge (q \to r)$	assumption
2.	$p \to q$	$\wedge e_1$ 1
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6.	r	
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5.	q	MP 2,4
6.	r	MP 3,5
7.	$p \to r$	$\to i$ 4-6
8.	$(p \to q) \wedge (q \to r) \to (p \to r)$	$\to i$ 1-7

Some exercises

Exercise 4.5

Consider the following sequent

$$p \wedge q \to r \vdash p \to (q \to r)$$

1. Is it a valid sequent?
2. Which of the steps in your derivation are deterministic?
3. Write down the "converse" of the sequent. Is it a valid sequent?

Some exercises

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1. Is it a valid sequent?
2. Which of the steps in your derivation are deterministic?
3. Write down the "converse" of the sequent. Is it a valid sequent?

Exercise 4.6

Using not more than 10 lines, prove that the following sequents are valid.

1. $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$
2. $\vdash (p \wedge q) \to q$
3. $p \to q \wedge r \vdash (p \to q) \wedge (p \to r)$

Exercise 4.5: Solution

$$p \wedge q \to r \vdash p \to (q \to r)$$

1.	$p \wedge q \to r$	premise
2.	p	assumption
3.	q	assumption
4.	$p \wedge q$	$\wedge i$ 2,3
5.	r	MP 1,4
6.	$q \to r$	$\to i$ 3-5
7.	$p \to (q \to r)$	$\to i$ 2-6

Exercise 4.5: Solution

$p \wedge q \to r \vdash p \to (q \to r)$

1.	$p \wedge q \to r$	premise
2.	p	assumption
3.	q	assumption
4.	$p \wedge q$	$\wedge i$ 2,3
5.	r	MP 1,4
6.	$q \to r$	$\to i$ 3-5
7.	$p \to (q \to r)$	$\to i$ 2-6

$p \to (q \to r) \vdash p \wedge q \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 5,4
7.	$p \wedge q \to r$	$\to i$ 2-6

Disjunction rules

If the ball is round then the ball is either round or green.

$$\frac{F}{F \vee G} \vee i_1$$

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- If $F \vee G$ is true and from BOTH F and G we are able to obtain H, then we can deduce H

Whether it's sunny or cloudy, I'll go for a walk — either way, I'm walking!

Disjunction rules (Contd)

Example: $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.		

Disjunction rules (Contd)

Example: $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.		

Disjunction rules (Contd)

Example: $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	assumption
4.		

Disjunction rules (Contd)

Example: $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
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3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.		

Disjunction rules (Contd)

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3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.		

Disjunction rules (Contd)

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5.	q	assumption
6.	r	MP 1,5
7.		

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4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6

Disjunction rules (Contd)

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5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7

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6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.		

Disjunction rules (Contd)

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3.	p	assumption
4.	$p \vee r$	$\vee i_1$ 3
5.	q	assumption
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.	$(p \vee q) \to (p \vee r)$	$\to i$ 2-8