CS228 : Logic for Computer Science

Module 3: Predicate Logic

Lecture 14: Natural Deduction for First-Order Logic

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IIT Bombay, India

Welcome back after the Midsem

Module 1: Propositional Logic

1. Syntax
2. Semantics
3. Formal proof systems
4. Normal Forms

Module 2: The Problem of Satisfiability

1. Propositional Resolution
2. DPLL
3. CDCL
4. SAT/SMT tool demo

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1. Syntax of predicate logic (Lecture 12)
2. Semantics of predicate logic (Lecture 13)
3. Today: a formal proof system for predicate logic (natural deduction)

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Let's warm up with a quick recap before diving into today's new material.

Predicate Logic: Alphabet and Syntax

Alphabet consists of

- Variable names: x, y, z
- Vocabulary
  - Function symbols: $f, g, h...$
  - Constants: a, b, c are just 0-ary function symbols.
  - Relation (Predicate) symbols: $P, Q, R, ...$
- Fixed symbols: $\wedge, \vee, \neg, \rightarrow, (,), \exists, \forall$

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- Formulas $\varphi ::= P(t_1, \ldots t_n) \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\forall x\varphi) \mid (\exists x\varphi)$

  where P is over set of predicates, $t_1, \ldots t_n$ are terms and x is a variable.

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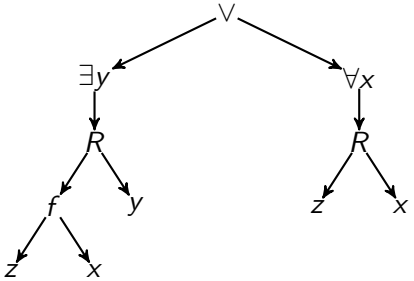
For ψ defined above: $\psi[f(z, x)/x] := \exists y R(f(z, x), y) \vee \forall x R(z, x)$

Substitution in FOL: Correct vs. Captured

Correct: $\psi[f(z,x)/x]$

$\exists y\, R(f(z,x),y) \vee \forall x\, R(z,x)$



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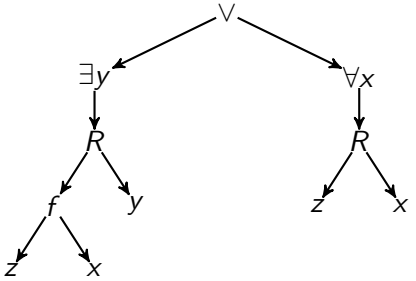

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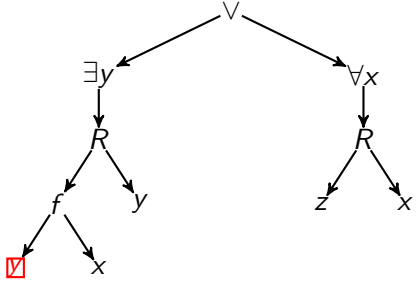

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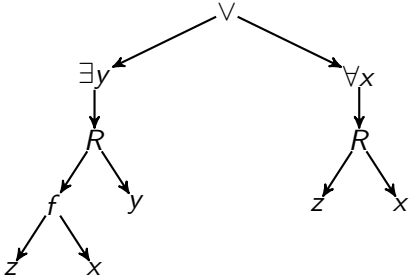

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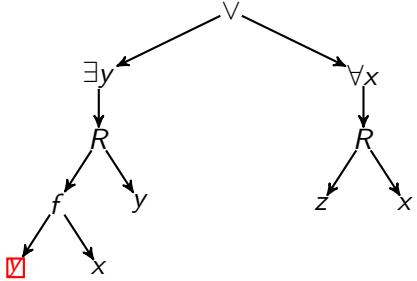

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- Fix: rename the bound y to a fresh y' first
- Rewrite ψ as $\exists y'\, R(x,y') \vee \forall x\, R(z,x)$ (same meaning, y is just a placeholder), *then* substitute.

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Today: can we nevertheless *prove* things about FOL formulas syntactically?

Topic 14.1

Natural Deduction for FOL

A Proof System for FOL

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- All proof rules for Propositional Logic carry over and remain sound for FOL.
- In addition, we need new rules for:
  - Equality ($=$)
  - Quantifiers ($\forall, \exists$)

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- If $y * (w + 2) = y * w + y * 2$; then $z \geq y * (w + 2)$ implies $z \geq y * w + y * 2$ (and vice versa)

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- Of course, this is applicable ONLY if t_1, t_2 are free for x in φ.
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Exercise 14.1 Show that the following sequents are valid.

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2. $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$

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- If from a fresh var x_0, we can prove $\varphi[x_0/x]$, then $\forall x \varphi$ holds.

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$$\frac{\forall x \varphi}{\varphi[t/x]}\ \forall x\ e$$

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ \varphi[x_0/x] \end{array}}}{\forall x \varphi}\ \forall x\ i$$

- All students did PS1 $\Rightarrow$ so did YOU!
- If $\forall x \varphi$ holds, then we can replace x in φ by any term t.
- Imp: side condition t is free for x in φ.
  - Consider $\varphi = \exists y(x < y)$ with vocab reals with usual smaller than relation.
  - $\forall x \varphi$ means for all real n, there exists real m bigger than n, true!
  - But $\varphi[y/x] = \exists y(y < y)$, which says there must exist a no. bigger than itself, false!
  - We want only sound rules!
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- Say I threw this tennis ball $\Rightarrow$ I can throw ANY tennis ball!
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- In proving φ, we didn't make an assumption on x.

Examples

Exercise 14.2 Prove that the following sequents are valid.

1. $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)$
2. $P(t), \forall x(P(x) \to \neg Q(x)) \vdash \neg Q(t)$ for any term t.

Exercise 14.2: Solution

1. $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)$
   1. $\forall x(P(x) \to Q(x))$ premise
   2.

Exercise 14.2: Solution

1. $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)$

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\forall x P(x)$	premise
3.		

Exercise 14.2: Solution

1. $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)$

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\forall x P(x)$	premise
3.	$P(x_0) \to Q(x_0)$	$\forall x$ e 1
4.		

Exercise 14.2: Solution

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4.	$P(x_0)$	$\forall x$ e 2
5.		

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6.	$\forall x Q(x)$	$\forall x$ i 3-5

(box scoped by fresh x_0)

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2. $P(t), \forall x(P(x) \to \neg Q(x)) \vdash \neg Q(t)$

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4.		

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1.	$P(t)$	premise
2.	$\forall x(P(x) \to \neg Q(x))$	premise
3.	$P(t) \to \neg Q(t)$	$\forall x$ e 2
4.	$\neg Q(t)$	MP 3,1

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Intuition: $\forall$ "generalizes" $\wedge$

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Intuition: $\forall$ "generalizes" $\wedge$

- Just like in $\wedge$-e, we could get F or G from $F \wedge G$, we can get $\varphi[t/x]$ for any term t (modulo side-condn!).
- Similarly $\wedge$-i had two premises, here $\forall x\ i$ has premise for any x_0.

Proof rules for Existential Quantifiers

- Can we lift this analogy and do the same for $\exists$ and $\vee$?

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$$\frac{F \vee G \qquad \boxed{\begin{array}{c} F \\ \vdots \\ H \end{array}} \qquad \boxed{\begin{array}{c} G \\ \vdots \\ H \end{array}}}{H}\ \vee e$$

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$$\frac{\exists x \varphi \qquad \begin{array}{c}[x_0\ \varphi[x_0/x]]\\ \vdots \\ \chi\end{array}}{\chi}\ \exists x\ e$$

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- Can we lift this analogy and do the same for $\exists$ and $\vee$?
- For $\vee$, $\vee$-i was easy! So what should be the rule for $\exists$-i? What about $\exists$-e?

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$$\frac{\exists x \varphi \qquad \boxed{\begin{array}{l} x_0\ \ \varphi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi}\ \exists x\ e$$

- To use $\exists x \varphi$, you should have a proof from any possible value for x.
- The box controls scope of x_0 and scope of assumption also.

Examples

Exercise 14.3 Show that the following sequents are valid.

1. $\forall x \varphi \vdash \exists x \varphi$
2. $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$
3. $\exists x P(x), \forall x \forall y(P(x) \to Q(y)) \vdash \forall y Q(y)$

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Solution for 1. $\forall x \varphi \vdash \exists x \varphi$

1. $\forall x \varphi$ premise
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Solution for 1. $\forall x \varphi \vdash \exists x \varphi$

1.	$\forall x \varphi$	premise	
2.	φ	$\forall x\, e\, 1$	(choose $t = x$ for both steps: $\varphi[x/x] = \varphi$, so this is legal)
3.	$\exists x \varphi$	$\exists x\, i\, 2$	

Exercise 14.3.2: Correct vs. Wrong

Correct

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\exists x P(x)$	premise
3.	$P(x_0)$	assumption
4.	$P(x_0) \to Q(x_0)$	$\forall x$ *e* 1
5.	$Q(x_0)$	MP 4,3
6.	$\exists x Q(x)$	$\exists x$ *i* 5
7.	$\exists x Q(x)$	$\exists x$ *e* 2, 3-6

Exercise 14.3.2: Correct vs. Wrong

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1.	$\forall x(P(x) \to Q(x))$	premise
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6.	$\exists x Q(x)$	$\exists x$ *i* 5
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Wrong!

1.	$\forall x(P(x) \to Q(x))$	premise
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4.	$P(x_0) \to Q(x_0)$	$\forall x$ *e* 1
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Exercise 14.3.2: Correct vs. Wrong

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The box on the right closes one line too early: line 6 ($Q(x_0)$) lets the fresh x_0 escape its own box before $\exists x$ *i* is applied outside – but χ in $\exists x$ *e* must not mention x_0!