CS228 : Logic for Computer Science

Module 3: Predicate Logic

Lecture 15: Soundness, Completeness, and Structures in FOL

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Topics Covered till date

Module 1: Propositional Logic

1. Syntax
2. Semantics
3. Formal proof systems
4. Normal Forms

Module 2: The Problem of Satisfiability

1. Propositional Resolution
2. DPLL
3. CDCL
4. SAT/SMT tool demo

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Module 3: Predicate Logic

1. Syntax of predicate logic (Lecture 12)
2. Semantics of predicate logic (Lecture 13)
3. Natural deduction for FOL (Lecture 14)
4. Today: is our proof system sound and complete, and just how powerful is FOL?

Quick Recap: Natural Deduction for FOL

- All proof rules for Propositional Logic carry over and remain sound for FOL.

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$$\frac{}{t = t}\ =_i \qquad \frac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]}\ =_e$$

Equality: reflexivity axiom, and substitute equals for equals (terms only, and only where free for x).

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$$\frac{\forall x \varphi}{\varphi[t/x]}\ \forall x\ e$$

$$\frac{\begin{array}{|l} x_0 \\ \vdots \\ \varphi[x_0/x] \end{array}}{\forall x \varphi}\ \forall x\ i$$

Universal quantifier: instantiate to any term t (free for x); or prove $\varphi[x_0/x]$ for a fresh x_0 inside a box.

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$$\frac{\varphi[t/x]}{\exists x \varphi}\ \exists x\ i \qquad \frac{\exists x \varphi \qquad \begin{array}{|ll} x_0 & \varphi[x_0/x] \\ & \vdots \\ & \chi \end{array}}{\chi}\ \exists x\ e$$

Existential quantifier: generalize $\varphi[t/x]$ to $\exists x \varphi$; or from $\exists x \varphi$ and a boxed proof of χ from fresh $x_0, \varphi[x_0/x]$, conclude χ (provided χ doesn't mention x_0).

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Side conditions on freshness/scope are important – violate them and rules become unsound!

Quick Recap: (Ungraded) Pop quiz!

Recall the exercise from last lecture: Prove the following sequents using ND for FOL:

1. $\forall x \varphi \vdash \exists x \varphi$

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1. $\forall x \varphi \vdash \exists x \varphi$
2. $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

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   Example: A factory's policy is "if any laptop in a batch has a faulty battery, recall every laptop in the batch." We're told some laptop has a faulty battery – so every laptop gets recalled!

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   ($P(x)$: x has a faulty battery; $Q(y)$: y gets recalled.)

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4. "Every prime number greater than 2 is odd. There exists a prime number greater than 2. So, there exists an odd number." First formalize this in FOL, then prove it using ND!

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4. "Every prime number greater than 2 is odd. There exists a prime number greater than 2. So, there exists an odd number." First formalize this in FOL, then prove it using ND!

Which rules would you use first? When do you need a fresh variable and when do you need a box?

First Question: $\forall x \varphi \vdash \exists x \varphi$

1. $\forall x \varphi \vdash \exists x \varphi$

1.	$\forall x \varphi$	premise
2.		

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1. $\forall x \varphi \vdash \exists x \varphi$

1.	$\forall x \varphi$	premise
2.	φ	$\forall x$ *e* 1
3.		

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1. $\forall x \varphi \vdash \exists x \varphi$

1.	$\forall x \varphi$	premise
2.	φ	$\forall x\ e\ 1$
3.	$\exists x \varphi$	$\exists x\ i\ 2$

(choose $t = x$ for both steps: $\varphi[x/x] = \varphi$, so this is legal)

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1. $\forall x(P(x) \to Q(x))$ premise
2.

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1. $\forall x(P(x) \to Q(x))$ premise
2. $\exists x P(x)$ premise
3.

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\exists x P(x)$	premise
3.	x_0 $P(x_0)$	assumption
4.		

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1.		$\forall x(P(x) \to Q(x))$	premise
2.		$\exists x P(x)$	premise
3.	x_0	$P(x_0)$	assumption
4.		$P(x_0) \to Q(x_0)$	$\forall x$ e 1
5.			

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1.		$\forall x(P(x) \to Q(x))$	premise
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3.	x_0	$P(x_0)$	assumption
4.		$P(x_0) \to Q(x_0)$	$\forall x$ e 1
5.		$Q(x_0)$	MP 4,3
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1.		$\forall x(P(x) \to Q(x))$	premise
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6.		$\exists x Q(x)$	$\exists x$ i 5
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1.		$\forall x(P(x) \to Q(x))$	premise
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3.	x_0	$P(x_0)$	assumption
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6.		$\exists x Q(x)$	$\exists x$ i 5
7.		$\exists x Q(x)$	$\exists x$ e 2, 3-6

Second Question: $\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\exists x P(x)$	premise
3.	x_0 $P(x_0)$	assumption
4.	$P(x_0) \to Q(x_0)$	$\forall x$ *e* 1
5.	$Q(x_0)$	MP 4,3
6.	$\exists x Q(x)$	$\exists x$ *i* 5
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Wrong!

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\exists x P(x)$	premise
3.	x_0 $P(x_0)$	assumption
4.	$P(x_0) \to Q(x_0)$	$\forall x$ *e* 1
5.	$Q(x_0)$	MP 4,3
6.	$Q(x_0)$	$\exists x$ *e* 2, 3-5
7.	$\exists x Q(x)$	$\exists x$ *i* 6

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1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\exists x P(x)$	premise
3.	x_0 $P(x_0)$	assumption
4.	$P(x_0) \to Q(x_0)$	$\forall x$ *e* 1
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5.	$Q(x_0)$	MP 4,3
6.	$Q(x_0)$	$\exists x$ *e* 2, 3-5
7.	$\exists x Q(x)$	$\exists x$ *i* 6

The box on the right closes one line too early: line 6 ($Q(x_0)$) lets the fresh x_0 escape its own box before $\exists x$ *i* is applied outside – but χ in $\exists x$ *e* must not mention x_0!

Third Question: $\exists x P(x), \forall x \forall y (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$

3. $\exists x P(x), \forall x \forall y (P(x) \rightarrow Q(y)) \vdash \forall y Q(y)$

1.	$\exists x P(x)$	premise
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1.	$\exists x P(x)$	premise
2.	$\forall x \forall y (P(x) \to Q(y))$	premise
3.	y_0	
4.		

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1.		$\exists x P(x)$	premise
2.		$\forall x \forall y (P(x) \rightarrow Q(y))$	premise
3.		y_0	
4.	x_0	$P(x_0)$	assumption
5.			

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3.	y_0	
4.	x_0 $P(x_0)$	assumption
5.	$\forall y (P(x_0) \to Q(y))$	$\forall x$ e 2
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3.	y_0		
4.	x_0	$P(x_0)$	assumption
5.		$\forall y (P(x_0) \to Q(y))$	$\forall x$ e 2
6.		$P(x_0) \to Q(y_0)$	$\forall y$ e 5
7.			

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3.	y_0	
4.	x_0 $P(x_0)$	assumption
5.	$\forall y (P(x_0) \to Q(y))$	$\forall x$ e 2
6.	$P(x_0) \to Q(y_0)$	$\forall y$ e 5
7.	$Q(y_0)$	MP 6,4
8.		

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2.		$\forall x \forall y (P(x) \to Q(y))$	premise
3.	y_0		
4.	x_0	$P(x_0)$	assumption
5.		$\forall y (P(x_0) \to Q(y))$	$\forall x$ e 2
6.		$P(x_0) \to Q(y_0)$	$\forall y$ e 5
7.		$Q(y_0)$	MP 6,4
8.		$Q(y_0)$	$\exists x$ e 1, 4-7
9.			

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3.	y_0	
4.	x_0 $P(x_0)$	assumption
5.	$\forall y (P(x_0) \to Q(y))$	$\forall x$ *e* 2
6.	$P(x_0) \to Q(y_0)$	$\forall y$ *e* 5
7.	$Q(y_0)$	MP 6,4
8.	$Q(y_0)$	$\exists x$ *e* 1, 4-7
9.	$\forall y Q(y)$	$\forall y$ *i* 3-8

(outer box scoped by fresh y_0; inner box scoped by fresh x_0, assuming $P(x_0)$)

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4.	$P(x_0) \to Q(x_0)$	$\forall x\, e$ 1
5.	$Q(x_0)$	MP 4,3
6.	$\exists x Q(x)$	$\exists x\, i$ 5
7.	$\exists x Q(x)$	$\exists x\, e$ 2, 3-6

Topic 15.1

Soundness, Completeness, and the Power of FOL

Soundness and completeness

Soundness: If $\varphi \vdash \psi$, then $\varphi \models \psi$

- We can only derive semantically correct proofs!
- Intuitively easy to see, again by structural induction.

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- Yes, but we won't stop to prove it here!

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- Is anything that is true, derivable?
- Yes, but we won't stop to prove it here!

So even checking whether $\vdash \varphi$ holds is undecidable – because validity itself ($\models \varphi$) is undecidable, and completeness tells us $\vdash$ and $\models$ coincide!

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Food for thought!

Topic 15.2

Structures and Expressiveness of FOL

Quick Recap: Semantics of Predicate Logic (Lecture 13)

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- Satisfaction $M, \alpha \models \varphi$: defined by structural induction on φ.

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  - Class of $\mathcal{V}$-structures defined by φ is just $\{M \mid M \models \varphi\}$
  - Such classes of structures that are defined by an FOL sentence are called FO-definable.
- We will see three commonly occurring examples of structures:
  - Numbers (we saw this already!)
  - Graphs
  - Databases

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Maybe we can check validity, but given the model/structure can we check if it satisfies a given formula? Model checking!