CS228 : Logic for Computer Science

Module 3: Predicate Logic

Lecture 16: Expressiveness and the Limits of FOL

Instructor: S. Akshay

IIT Bombay, India

Topics Covered till date

Module 1: Propositional Logic

1. Syntax
2. Semantics
3. Formal proof systems
4. Normal Forms

Module 2: The Problem of Satisfiability

1. Propositional Resolution
2. DPLL
3. CDCL
4. SAT/SMT tool demo

Topics Covered till date

Module 1: Propositional Logic

1. Syntax
2. Semantics
3. Formal proof systems
4. Normal Forms

Module 2: The Problem of Satisfiability

1. Propositional Resolution
2. DPLL
3. CDCL
4. SAT/SMT tool demo

Module 3: Predicate Logic

1. Syntax of predicate logic (Lecture 12)
2. Semantics of predicate logic (Lecture 13)
3. Natural deduction for FOL (Lecture 14)
4. Soundness, completeness, and structures in FOL (Lecture 15)
5. Today: Expressiveness and the limits of FOL

Quick Recap: First-order Definable Structures

- Recall: for a sentence φ (no free variables), $\{M \mid M \models \varphi\}$ – the class of structures it defines – is called FO-definable.

Quick Recap: First-order Definable Structures

- Recall: for a sentence φ (no free variables), $\{M \mid M \models \varphi\}$ – the class of structures it defines – is called FO-definable.
- Already saw: Numbers as FO structures, $\mathfrak{N} = (\mathbb{N}, 0, 1, \times, +, <, =)$ and $\mathfrak{R} = (\mathbb{R}, 0, 1, \times, +, <, =)$.

Quick Recap: First-order Definable Structures

- Recall: for a sentence φ (no free variables), $\{M \mid M \models \varphi\}$ – the class of structures it defines – is called FO-definable.
- Already saw: Numbers as FO structures, $\mathfrak{N} = (\mathbb{N}, 0, 1, \times, +, <, =)$ and $\mathfrak{R} = (\mathbb{R}, 0, 1, \times, +, <, =)$.
  - E.g.for $\psi = \forall x \exists y (x = y \times y \times y)$, $\mathfrak{R} \models \psi$ (every real has a cube root), but $\mathfrak{N} \not\models \psi$; and infinitely many primes (Exercise 15.1).

Quick Recap: First-order Definable Structures

- Recall: for a sentence φ (no free variables), $\{M \mid M \models \varphi\}$ – the class of structures it defines – is called FO-definable.
- Already saw: Numbers as FO structures, $\mathfrak{N} = (\mathbb{N}, 0, 1, \times, +, <, =)$ and $\mathfrak{R} = (\mathbb{R}, 0, 1, \times, +, <, =)$.
  - E.g.for $\psi = \forall x \exists y (x = y \times y \times y)$, $\mathfrak{R} \models \psi$ (every real has a cube root), but $\mathfrak{N} \not\models \psi$; and infinitely many primes (Exercise 15.1).
- Today: Databases and Graphs as FO structures, then just how far this expressive power actually goes.

Topic 16.1

Structures and Expressiveness of FOL

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Example of class of databases definable in FOL:
$\forall x \forall y \forall z \, (Record(x, y, z) \leftrightarrow (DOB(x, y) \wedge Class(x, z)))$

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Example of class of databases definable in FOL:

$\forall x \forall y \forall z \, (Record(x, y, z) \leftrightarrow (DOB(x, y) \wedge Class(x, z)))$: Record is the join of DOB and Class.

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Example of class of databases definable in FOL:
$\forall x \forall y \forall z \, (Record(x, y, z) \leftrightarrow (DOB(x, y) \wedge Class(x, z)))$: Record is the join of DOB and Class.
Example Query: $\varphi(x) := \exists y \exists z (DOB(x, y) \wedge After(y, \text{“2016”}) \wedge Class(x, z) \wedge Primary(z))$

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Example of class of databases definable in FOL:
$\forall x \forall y \forall z \, (Record(x, y, z) \leftrightarrow (DOB(x, y) \wedge Class(x, z)))$: Record is the join of DOB and Class.
Example Query: $\varphi(x) := \exists y \exists z (DOB(x, y) \wedge After(y, \text{“2016”}) \wedge Class(x, z) \wedge Primary(z))$
Defines set of students born after 2016, and studying in a primary class.

Relational Databases as FO structures

A school's database has tables like `DOB` (a student's date of birth), `Class` (which class a student is in), and `Primary` (which classes are primary-school classes)

DOB	
Student	Year
s_1	2010
s_2	2018

Class	
Student	Class
s_1	c_2
s_2	c_1

Primary
c_1

Relational Database D (a $\mathcal{V}$-structure/model)

- U^D: set of (possibly differently typed) data items
- Vocab $\mathcal{V} : \{P_1, \ldots P_k, =\}$, where P_i is k_i-ary pred corr to i^{th} table in database with k_i cols.
- Interpret: For $a_1, \ldots a_{k_i} \in U^D$, $P_i(a_1, \ldots a_{k_i}) = \top$ iff $(a_1, \ldots a_{k_i})$ is a row of i^{th} table.

Example of class of databases definable in FOL:

$\forall x \forall y \forall z \, (Record(x, y, z) \leftrightarrow (DOB(x, y) \wedge Class(x, z)))$: Record is the join of DOB and Class.

Example Query: $\varphi(x) := \exists y \exists z (DOB(x, y) \wedge After(y, \text{“2016”}) \wedge Class(x, z) \wedge Primary(z))$

Defines set of students born after 2016, and studying in a primary class.

Model checking: Does a given Database satisfy a given property?

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0, 1\}$, including the empty word ϵ.

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \dots a_k \cdot^W b_1 \dots b_n = a_1 \dots a_k b_1 \dots b_n$.

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \ldots a_k \cdot^W b_1 \ldots b_n = a_1 \ldots a_k b_1 \ldots b_n$.
- Interpret $\leq^W$: the prefix ordering – $s_1 \leq^W s_2$ iff $\exists s_3$ such that $s_1 \cdot^W s_3 = s_2$

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \ldots a_k \cdot^W b_1 \ldots b_n = a_1 \ldots a_k b_1 \ldots b_n$.
- Interpret $\leq^W$: the prefix ordering – $s_1 \leq^W s_2$ iff $\exists s_3$ such that $s_1 \cdot^W s_3 = s_2$

E.g. 011 is a prefix of 011001 and of 011, but 010 is neither.

Pop Quiz. For each formula φ below, write its English meaning and argue whether it is true in W – i.e., does $W \models \varphi$ hold?

- **Q1.** $\exists y \, \forall x \, (y \leq x)$

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \ldots a_k \cdot^W b_1 \ldots b_n = a_1 \ldots a_k b_1 \ldots b_n$.
- Interpret $\leq^W$: the prefix ordering – $s_1 \leq^W s_2$ iff $\exists s_3$ such that $s_1 \cdot^W s_3 = s_2$

E.g. 011 is a prefix of 011001 and of 011, but 010 is neither.

Pop Quiz. For each formula φ below, write its English meaning and argue whether it is true in W – i.e., does $W \models \varphi$ hold?

- **Q1.** $\exists y \, \forall x \, (y \leq x)$
- **Q2.** $\forall x \, \exists y \, (y \leq x)$

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \ldots a_k \cdot^W b_1 \ldots b_n = a_1 \ldots a_k b_1 \ldots b_n$.
- Interpret $\leq^W$: the prefix ordering – $s_1 \leq^W s_2$ iff $\exists s_3$ such that $s_1 \cdot^W s_3 = s_2$

E.g. 011 is a prefix of 011001 and of 011, but 010 is neither.

Pop Quiz. For each formula φ below, write its English meaning and argue whether it is true in W – i.e., does $W \models \varphi$ hold?

- **Q1.** $\exists y \, \forall x \, (y \leq x)$
- **Q2.** $\forall x \, \exists y \, (y \leq x)$
- **Q3.** $\forall x \, \forall y \, \forall z \, \big((x \leq y) \rightarrow (x \cdot z \leq y \cdot z)\big)$

Words as FO structures

A word structure W (a $\mathcal{V}$-structure/model)

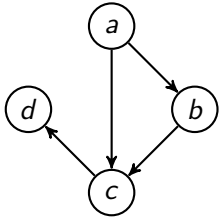
- U^W: set of all finite binary words (strings) over $\{0,1\}$, including the empty word ϵ.
- Vocab $\mathcal{V} : \{\cdot, \leq, =\}$, where $\cdot$ is a binary function symbol and $\leq$ a binary predicate symbol.
- Interpret $\cdot^W$: concatenation, e.g. $0110 \cdot^W 1110 = 01101110$; in general $a_1 \ldots a_k \cdot^W b_1 \ldots b_n = a_1 \ldots a_k b_1 \ldots b_n$.
- Interpret $\leq^W$: the prefix ordering – $s_1 \leq^W s_2$ iff $\exists s_3$ such that $s_1 \cdot^W s_3 = s_2$

E.g. 011 is a prefix of 011001 and of 011, but 010 is neither.

Pop Quiz. For each formula φ below, write its English meaning and argue whether it is true in W – i.e., does $W \models \varphi$ hold?

- **Q1.** $\exists y \, \forall x \, (y \leq x)$
- **Q2.** $\forall x \, \exists y \, (y \leq x)$
- **Q3.** $\forall x \, \forall y \, \forall z \, \big((x \leq y) \rightarrow (x \cdot z \leq y \cdot z)\big)$
- **Q4.** $\neg \exists x \, \forall y \, \big((x \leq y) \rightarrow (y \leq x)\big)$

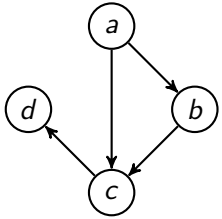
Graphs as FO structures



A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices

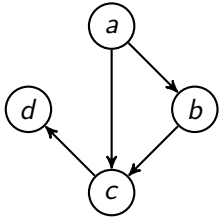
Graphs as FO structures



A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation

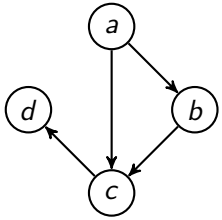
Graphs as FO structures



A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Graphs as FO structures



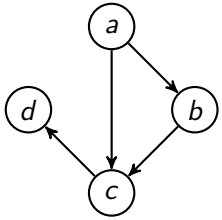
A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Examples of graphs in FOL

- What does $\forall x \forall y(\neg(x = y) \rightarrow E(x, y))$ represent?

Graphs as FO structures



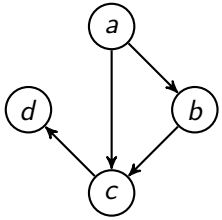
A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Examples of graphs in FOL

- What does $\forall x \forall y(\neg(x = y) \rightarrow E(x, y))$ represent? (Infinite) Class of cliques!

Graphs as FO structures



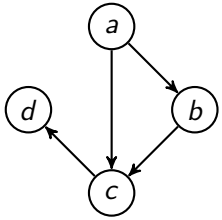
A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Examples of graphs in FOL

- What does $\forall x \forall y(\neg(x = y) \rightarrow E(x, y))$ represent? (Infinite) Class of cliques!
- **Exercise 15.2** Give a FOL formula to define class of all graphs with no cycles of length 3.

Graphs as FO structures



A graph G (a $\mathcal{V}$-structure/model)

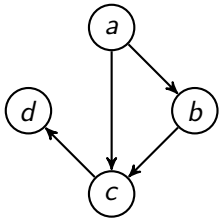
- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Examples of graphs in FOL

- What does $\forall x \forall y(\neg(x = y) \rightarrow E(x, y))$ represent? (Infinite) Class of cliques!
- **Exercise 15.2** Give a FOL formula to define class of all graphs with no cycles of length 3.
  - A directed 3-cycle is $x \rightarrow y \rightarrow z \rightarrow x$ for three *distinct* vertices:

$$\exists x \exists y \exists z \left(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z) \wedge E(x, y) \wedge E(y, z) \wedge E(z, x)\right)$$

Graphs as FO structures



A graph G (a $\mathcal{V}$-structure/model)

- U^G is a set of vertices
- Vocab $\mathcal{V} : \{E, =\}$ where E is the (binary) edge relation
- Interpret: For $a, b \in U^G$, $E^G(a, b) = \top$ iff there is a(n directed) edge from vertex a to vertex b in G.

Examples of graphs in FOL

- What does $\forall x \forall y(\neg(x = y) \rightarrow E(x, y))$ represent? (Infinite) Class of cliques!
- **Exercise 15.2** Give a FOL formula to define class of all graphs with no cycles of length 3.
  - A directed 3-cycle is $x \rightarrow y \rightarrow z \rightarrow x$ for three *distinct* vertices:

$$\exists x \exists y \exists z \left(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z) \wedge E(x, y) \wedge E(y, z) \wedge E(z, x)\right)$$

  - So "no 3-cycles" is the negation of that:

$$\forall x \forall y \forall z \left(\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z) \wedge E(x, y) \wedge E(y, z) \rightarrow \neg E(z, x)\right)$$

Power of FOL

- FOL seems all powerful!

Power of FOL

- FOL seems all powerful!
- Let us turn to a simpler question:
  - Given u, v in a directed graph, does there exist a finite path from u to v?
  - i.e., Give a FOL formula φ, which has u, v as free variables and E as the binary edge relation, such that
    - $G \models \varphi$ iff there is a path in G from u to v

Power of FOL

- FOL seems all powerful!
- Let us turn to a simpler question:
  - Given u, v in a directed graph, does there exist a finite path from u to v?
  - i.e., Give a FOL formula φ, which has u, v as free variables and E as the binary edge relation, such that
    - $G \models \varphi$ iff there is a path in G from u to v
    - So every graph G (with u, v assigned to specific vertices) satisfying φ must have a directed path from u to v

Power of FOL

- FOL seems all powerful!
- Let us turn to a simpler question:
  - Given u, v in a directed graph, does there exist a finite path from u to v?
  - i.e., Give a FOL formula φ, which has u, v as free variables and E as the binary edge relation, such that
    - $G \models \varphi$ iff there is a path in G from u to v
    - So every graph G (with u, v assigned to specific vertices) satisfying φ must have a directed path from u to v
    - $u = v \vee E(u, v) \vee \exists x(E(u, x) \wedge E(x, v)) \vee \ldots$ is infinite! So not correct...

Power of FOL

- FOL seems all powerful!
- Let us turn to a simpler question:
  - Given u, v in a directed graph, does there exist a finite path from u to v?
  - i.e., Give a FOL formula φ, which has u, v as free variables and E as the binary edge relation, such that
    - $G \models \varphi$ iff there is a path in G from u to v
    - So every graph G (with u, v assigned to specific vertices) satisfying φ must have a directed path from u to v
    - $u = v \vee E(u, v) \vee \exists x(E(u, x) \wedge E(x, v)) \vee \ldots$ is infinite! So not correct...

Surprise! There is no FOL formula that can express this!

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!

Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)
- By completeness, $\Gamma \vdash \bot$ is valid, in other words there is a proof in ND for this.

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)
- By completeness, $\Gamma \vdash \bot$ is valid, in other words there is a proof in ND for this.
- This proof is a finite set of lines, so uses only finitely many premises Δ from Γ.

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)
- By completeness, $\Gamma \vdash \bot$ is valid, in other words there is a proof in ND for this.
- This proof is a finite set of lines, so uses only finitely many premises Δ from Γ.
- But then $\Delta \vdash \bot$ is also valid.

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)
- By completeness, $\Gamma \vdash \bot$ is valid, in other words there is a proof in ND for this.
- This proof is a finite set of lines, so uses only finitely many premises Δ from Γ.
- But then $\Delta \vdash \bot$ is also valid.
- By soundness, we get $\Delta \models \bot$..

Compactness of FOL

Turns out there is a very interesting property of FOL that implies this and many more results...

Compactness theorem

Let Γ be a (possibly infinite) set of FOL sentences. If all finite subsets of Γ are satisfiable/consistent, then so is Γ.

Corollary: If Γ is unsatisfiable/inconsistent, then there is a finite subset of Γ that is already unsat!
Proof of Theorem follows from completeness of FOL!

- If Γ is unsatisfiable, then $\Gamma \models \bot$ holds (why?)
- By completeness, $\Gamma \vdash \bot$ is valid, in other words there is a proof in ND for this.
- This proof is a finite set of lines, so uses only finitely many premises Δ from Γ.
- But then $\Delta \vdash \bot$ is also valid.
- By soundness, we get $\Delta \models \bot$.. thus we have a finite subset of Γ that is already unsat, as required!

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c'

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3** hint:

$$\psi_n := \exists x_1 \ldots x_{n-1}(E(c, x_1) \wedge E(x_1, x_2) \ldots \wedge E(x_{n-1}, c'))$$

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.
- All formulas in Δ are sentences and it is unsatisfiable. (why?)

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.
- All formulas in Δ are sentences and it is unsatisfiable. (why?)
- But every finite subset of Δ is satisfiable! (why?)

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.
- All formulas in Δ are sentences and it is unsatisfiable. (why?)
- But every finite subset of Δ is satisfiable!(why?)
- By compactness, Δ is satisfiable! Contradiction.

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.
- All formulas in Δ are sentences and it is unsatisfiable. (why?)
- But every finite subset of Δ is satisfiable!(why?)
- By compactness, Δ is satisfiable! Contradiction.

So how do we increase expressivity?

Reachability is not expressible in FOL

Theorem: Reach is not expressible in FOL

- Suppose not. Then let ψ be the formula and let c, c' be consts.
- Let ψ_n be the formula expressing that there is a path of length n from c to c' **Exercise 15.3**
- Let $\Delta = \{\neg\psi_i \mid i \geq 0\} \cup \{\psi[c/u][c'/v]\}$.
- All formulas in Δ are sentences and it is unsatisfiable. (why?)
- But every finite subset of Δ is satisfiable!(why?)
- By compactness, Δ is satisfiable! Contradiction.

So how do we increase expressivity?
Quantify over sets!– higher order logic!!

Another corollary

Using compactness to show the following theorem.

Exercise 15.4

Let ψ be a FOL sentence s.t. $\forall n \geq 1$, there is a $\mathcal{V}$-structure M_n with $\geq n$ elements in its universe, such that $M_n \models \psi$, then there exists a $\mathcal{V}$-structure M with infinitely many elements in universe s.t. $M \models \psi$.