CS228 : Logic for Computer Science

Module 4: Hoare Logic and Program Verification

Lecture 19: Introduction to Hoare Logic

Instructor: S. Akshay

IIT Bombay, India

Logistics

Quiz 2

- **Friday, Oct 23rd, 6pm**
- Syllabus:
  - Module 3 (Predicate Logic/FOL): all of Lectures 12–18
  - Module 4 (Hoare logic): from Lecture 19 onwards, covered till Thursday, Oct 15th

Logistics

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SMT solver demo

- Postponed to next week.

Topics Covered till date

Module 1: Propositional Logic

1. Syntax
2. Semantics
3. Formal proof systems
4. Normal Forms

Module 2: The Problem of Satisfiability

1. Propositional Resolution
2. DPLL
3. CDCL
4. SAT/SMT tool demo

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Module 4: Hoare Logic and Program Verification

Today: Introduction to Hoare logic

Recap: Hoare Logic and Program Verification



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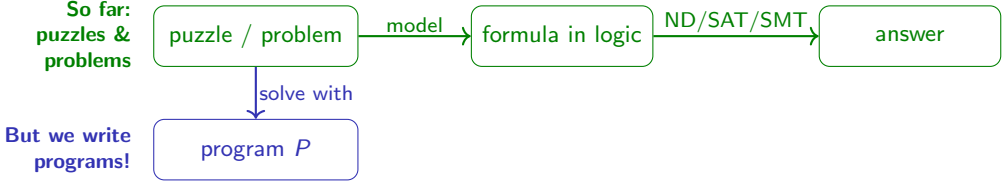

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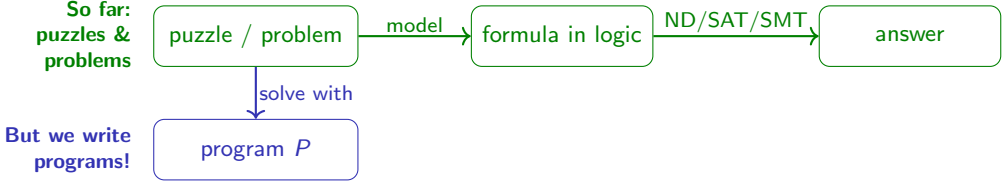

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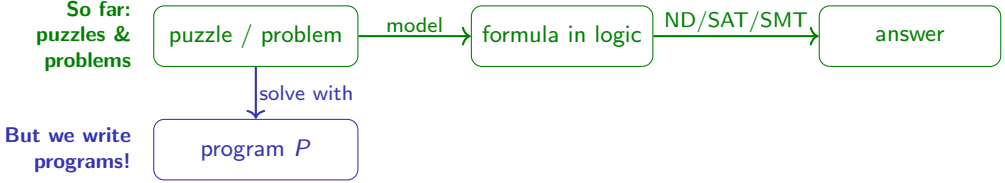

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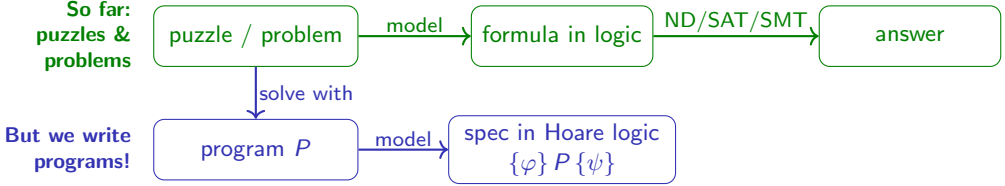

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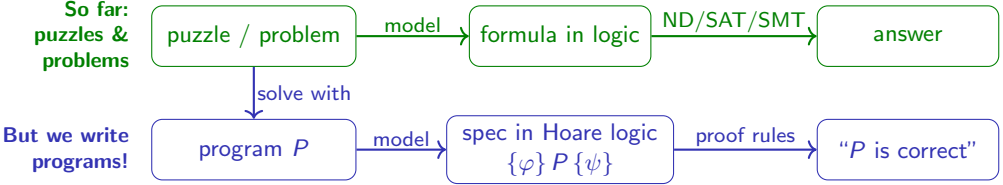

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- A logic of programs, Hoare logic, built on top of FOL!
- In this module: for *simple* imperative programs, we'll see how to *state* properties in logic (pre/postconditions) and *prove* them using proof rules.
- Real program verifiers (e.g., Dafny) largely automate the proof step using SMT solvers!

Consider a simple program

What does this program do? Take x as input and y as output.

	$y := 1;$
	$z := 0;$
	while $(z \neq x)\{$
inp x	$\quad z := z + 1;$
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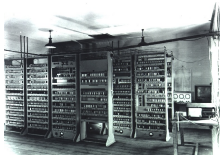

EDSAC (1948)

In 1949, at a Cambridge conference, Turing asked *"How can one check a routine in the sense of making sure that it is right?"* His example was a factorial program, with assertions for correctness and a quantity that decreases and vanishes for termination – exactly our two statements!



Alan Turing

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- So, as for logic, we need
  - Syntax: precisely which strings are programs? (a grammar)
  - Semantics: precisely what does a program do? (how it changes the state)
- Where have we done this before? Propositional logic (Lectures 1–2) and predicate logic (Lectures 12–13)!

Topic 19.1

A Simple Programming Language

A simple (imperative) programming language

Integer expressions

Variables $x, y, z, \ldots$, Numerals $0, 1, 2, \ldots, -1, -2, \ldots$, basic operations: addition ($+$), subtraction ($-$), multiplication (*), negation.

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$$E ::= n \mid x \mid (-E) \mid (E + E) \mid (E - E) \mid (E * E)$$

n is an integer, x a variable. We drop parentheses using the usual binding conventions: negation, then multiplication, then subtraction and addition.

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- **Exercise 19.2** Which of these does the grammar generate?
  1. $(4 + (x - 3))$
  2. $(x + ((x * x) * (y - (5 + z))))$
  3. $x + y * 2$

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  3. $x + y * 2$ Yes, because of binding conventions: it abbreviates $(x + (y * 2))$.

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Boolean expressions

Why do we need them?

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- Note that this builds on integer expressions
- As usual we can define other operators using these.
- **Exercise 19.3** Define $E_1 \neq E_2, E_1 < E_2$ using the above.

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Commands, i.e., programs

$$C ::= (x := E) \mid C; C \mid \textbf{if } B \ \{C\} \textbf{ else } \{C\} \mid \textbf{while } B \ \{C\}$$

where { and } mark the start and end of blocks of code

Syntax and semantics of a simple program

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  - Intuitively, program evaluates integer expression E in current state and overwrites current value stored in x with the value computed by the evaluation.

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- $C; C$ is sequential composition.
  - $C_1; C_2$ means execute C_1 in current state and if it terminates, then execute C_2 from the state at which it terminated.

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- **while** $B \ \{C\}$ evaluates B at current state. If B evaluates false the command terminates. Else, C is executed. When C terminates we re-evaluate B with updated state and continue. Thus, we repeatedly evaluate C as long as B is true.

The simple programming language

Exercise 19.4

1. Write a program in the above syntax to
   1.1 compute 2^n from n.
   1.2 find greatest y whose square is less than x
2. What features of a common imperative (i.e., state based) programming language like C, java, are not captured? Give 4 examples. Is it easy to modify the language to have these features?

Topic 19.2

Specifying Programs: Hoare Triples

Hoare Logic, sometimes called Floyd–Hoare Logic

Main idea: Use logic to capture what must be true before and after a command!

In general, a **Hoare triple** $\{\varphi\}\, P\, \{\psi\}$ is a formula of Hoare logic:

- φ and ψ are formulas in a base logic, e.g., predicate logic, Presburger arithmetic, etc. In our case, we will use first order logic over integers.

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Robert Floyd



Tony Hoare

Floyd 1967, Hoare 1969

Turing Awards: 1978, 1980

States and formulas

States and Formulas

- A state of the program is an environment, i.e., a function α that assigns to each variable x an integer $\alpha(x)$ (as in Lecture 13).
- Assertions φ, ψ are formulas of first order logic over the vocabulary $\mathcal{V}$ with the numerals as constants, function symbols $-$ (unary) and $+, *, -$ (binary), and predicate symbols $\leq$ and $=$.
- Let $M_{\mathbb{Z}} = (\mathbb{Z}, (\ldots, -1, 0, 1, \ldots, -, +, *, \leq))$ be the $\mathcal{V}$-structure in which every symbol has its usual meaning. We write $\alpha \models \varphi$ for $M_{\mathbb{Z}}, \alpha \models \varphi$.

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Exercise 19.5

Let α be a state where $\alpha(x) = -2, \alpha(y) = 5, \alpha(z) = -1$. Which of the following hold?

(a) $\alpha \models \neg(x + y < z)$

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(a) holds: $x + y = 3$ is not less than $z = -1$. (b) fails: for $u = 6$, $y * z = -5 < -6 = u * z$ is false.

Examples of Hoare Triples

Some examples of correct Hoare triples

- $\{(n \geq 0) \wedge (n * n > 20)\}\ \mathrm{n} := \mathrm{n} + 1; \mathrm{m} := \mathrm{m} * \mathrm{m}\ \{\neg(m = 50)\}$

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Exercise 19.6

Give 3 "correct" and 3 "incorrect" Hoare triples.

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How do we formalize the notion of correctness?

Correctness and semantics of Hoare Triples

Partial correctness

A triple $\{\varphi\}P\{\psi\}$ is satisfied under partial correctness, denoted $\models_{par} \{\varphi\}P\{\psi\}$ if, for all states that satisfy φ, the state resulting from P's execution satisfies the postcondition ψ, provided P actually terminates.

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Examples: Recall the program $\mathbf{Fac}_1$ from earlier,

- $\{\top\}\ \mathbf{Fac}_1\ \{y = x!\}$ satisfies partial correctness, but not total correctness.
- $\{x \geq 0\}\ \mathbf{Fac}_1\ \{y = x!\}$ satisfies partial AND total correctness.

Examples

Fac$_1$ *inp* x *out* y	$y := 1;$ $z := 0;$ **while** $(z \neq x)\{$ $\quad z := z + 1;$ $\quad y := y * z;$ $\}$

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Unk$_1$ *inp* x *out* y	$a := x + 1;$ **if** $(a - 1 = 0)\{$ $\quad y := 1;$ $\}$ **else** $\{$ $\quad y := a;$ $\}$

- $\{\top\}$ **Fac$_1$** $\{y = x!\}$ satisfies partial correctness, but NOT total correctness.
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Exercise 19.7

1. What does Unk_1 do? Take x as input and y as output.
2. Write a Hoare triple that captures the spec. Does it hold under partial/total correctness?

Program variables and logical variables

Fac$_1$ *inp* x *out* y	$y := 1;$ $z := 0;$ **while** $(z \neq x)\{$ $\quad z := z + 1;$ $\quad y := y * z;$ $\}$

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- **Fac**$_2$ is another program that computes the factorial.
- $\{x \geq 0\}$ **Fac**$_1$ $\{y = x!\}$ satisfies partial and total correctness.
- Qn: What about $\{x \geq 0\}$ **Fac**$_2$ $\{y = x!\}$?

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- we obtain $\{x = x_0 \wedge x \geq 0\}$ **Fac**$_2$ $\{y = x_0!\}$

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- How do we correct this? What is the correct specification? Need to remember initial value!
- we obtain $\{x = x_0 \wedge x \geq 0\}$ **Fac**$_2$ $\{y = x_0!\}$ x_0 is called a logical variable.