CS310 : Automata Theory 2019

Lecture 24: Turing Machines and Computability

Instructor: S. Akshay

IITB, India

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Turing Machines

Definition

A Turing Machine is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

1. Q is a finite set of states,
2. Σ is a finite input alphabet,
3. Γ is a finite tape alphabet where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. q_0 is the start state,
5. q_{acc} is the accept state,
6. q_{rej} is the reject state,
7. $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ is the transition function.

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For a $q \in Q$, $a \in \Gamma$ if $\delta(q, a) = (q', b, L)$, then

- q' is the new state of the machine,
- b is the letter which replaces a on the tape,
- the head moves to Left of the current position.

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- left move: $u\ a\ q_i\ b\ v$ yields $u\ q_j\ a\ c\ v$ if $\delta(q_i, b) = (q_j, c, L)$
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- left-end: $q_i\ b\ v$ yields (1) $q_j\ c\ v$ if transition is left moving or (2) $c\ q_j\ v$ if it is right moving
- right-end: assume that $u\ a\ q_i$ is the same as $u\ a\ q_i \sqcup$ as tape has blanks to the right.

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- Language of TM M, denoted $L(M)$, is the set of strings accepted by it.

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Every decidable language is Turing recognizable, but is the converse true?

Another example of a Turing Machine

A TM as a computer of integer functions

- Compute $\max\{m-n, 0\}$: construct TM M that starts with 0^m10^n and halts with $0^{\max\{m-n,0\}}$ on tape.

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Same as defining $B = \{a^m b^n c^{\max\{m-n,0\}} \mid m, n \geq 0\}$ and asked for a TM which accepts language B.

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- The repetition ends if
  1. searching right for 0, M encounters a blank.
     - Then all n 0's in 0^m10^n have been changed to 1 and $n+1$ of m 0's changed to blank. M replaces $n+1$ 1's by a 0 and n blanks leaving $m-n$ 0's on the tape.

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  2. beginning the cycle, M cannot find a 0 to change to a blank, because first m 0's have already been changed.
     - Then $n \geq m$, so $\max\{m-n,0\} = 0$. M replaces remaining 1's, 0's by blanks.

More examples/exercises

Exercises

1. $C = \{0^{2^n} \mid n \geq 0\}$, i.e., all strings of 0's whose length is a power of 2.
   - Give a high level description of a TM that accepts C.
   - Give the formal TM construction and state-diagram.

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   - Give a high level description of a TM that accepts C.
   - Give the formal TM construction and state-diagram.
2. $D = \{a^i b^j c^k \mid i * j = k \text{ and } i, j, k \geq 1\}$. Give a (high level description of) a TM that accepts D.

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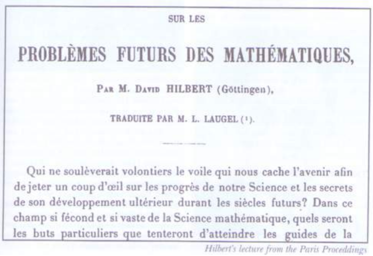
- So then again, why Turing? Why is the top-most nobel-equivalent award in computer science called Turing award and not Bill Gates award?

Turing Machines: a lesson from history



SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (¹).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d'atteindre les guides de la

Hilbert's lecture from the Paris Proceedings

Hilbert's problems

- In 1900, David Hilbert listed out 23 problems as challenges for 20th century at the Int. Cong. of Mathematicians in Paris.
- 10th problem: Devise an algorithm (a process doable using a finite no. of operations) to test if a given polynomial has integral roots.

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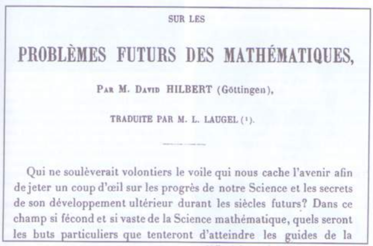

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- 10th problem: Devise an algorithm (a process doable using a finite no. of operations) to test if a given polynomial has integral roots.
- Now we know that no such algorithm exists. But how to prove this without a mathematical definition of an algorithm?

The Entscheidungsproblem

A challenge... in 1928

Is there an algorithm to decide whether any given statement is provable from the axioms using the rules of logic?

The Church-Turing Thesis

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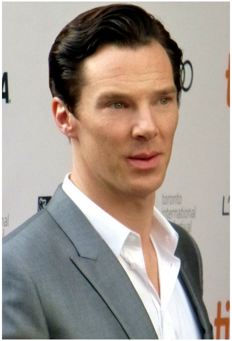
(1903–1995)



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- Thus, Turing machines capture our intuitive idea of computation.
- Indeed, we are more interested in algorithms themselves. Once we believe that TMs precisely capture algorithms, we will shift our focus to algorithms rather than low-level descriptions of TMs.

More on the Church-Turing Thesis

Lambda Calculus

- In 1936, Church introduced Lambda Calculus as a formal description of all computable functions.
- Independently, Turing had introduced his A-machines in 1936 too.
- Turing also showed that his A-machines were equivalent to Lambda Calculus of Church.

- So, can a Turing machine do everything? In other words are there algorithms to solve every question. Godel's incompleteness result asserts otherwise.
- If there is TM solving a problem, does there exist an equivalent TM that halts?

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- But Turing also worked on ideas and concepts that later made profound impact in AI and cryptography.
- "Father" of modern computers and computer science.