CS310 : Automata Theory 2019

Lecture 27: Turing Machines and undecidability

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Relationship among languages

$$\text{Regular} \subsetneq \text{Decidable} \underset{?}{\subsetneq} \text{Turing recognizable} \underset{?}{\subsetneq} \text{All languages}$$

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- Set of all languages over Σ is the set of subsets of S and is therefore uncountable Why? - recall Cantor from Discrete Structure's course.
- So for some such language, there must be no accepting TM.

Diagonalization

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- Proving existence just needs one to exhibit a function
- But how do we prove non-existence? *Try contradiction.*

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Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	✓	×	×	×	...
$f(1)$	✓	×	✓	✓	...
$f(2)$	×	×	×	×	...
$f(3)$	×	✓	×	✓	...

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$f(1)$	$\checkmark$	$\not\times$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\not\times$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\not\checkmark$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.

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$f(3)$	×	✓	×	~~✓~~ ×	...

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- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- S and $f(j)$ differ at position j, for any j.
- Thus, $S \neq f(j)$ for all $j \in \mathbb{N}$, which is a contradiction! □

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Theorem

L^A_{TM} is undecidable.

Proof of undecidability

Suppose $L^A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ was decidable.

1. Let H be the deciding TM: on input $\langle M, w\rangle$,

$$H(\langle M, w\rangle) = \begin{cases} \mathit{accept} & \text{if } M \text{ accepts } w \\ \mathit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

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Proof of undecidability

Diagonalization in the above argument

Enumerate Turing machines in the y-axis and their encodings in the x-axis.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	...	$\langle D \rangle$	...
M_1	$\overline{accept}$	reject	accept	...	accept	...
M_2	accept	$\overline{accept}$	accept	...	accept	...
M_3	reject	reject	$\overline{reject}$	...	reject	...
$\vdots$			$\vdots$		$\vdots$	
$D = M_i$	reject	reject	accept	...	$\overline{(??)}$	...
$\vdots$			$\vdots$		$\vdots$	

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So, what about $\overline{L^A_{TM}}$?