CS310 : Automata Theory 2019

Lecture 28: Reductions

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Recap of previous lecture

Regular $\subsetneq$ Decidable $\subsetneq$ Recursively Enumerable $\subsetneq$ All languages

DFA/NFA $<$ Algorithms/Halting TM $<$ Semi-algorithms/TM

Properties

1. There exist languages that are not R.E.
2. There exist languages that are R.E but are undecidable.
   Eg. universal TM lang $L^A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
3. Decidable languages are closed under complementation.
4. L is decidable iff L is $R.E$ and $\overline{L}$ is also R.E.

The halting problem

The halting problem for Turing Machines is undecidable

Does a given Turing machine halt on a given input?

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Proof: Suppose there exists TM H deciding L_{TM}^{HALT}, then construct a TM D s.t., on input $\langle M, w\rangle$:

- runs TM H on input $\langle M, w\rangle$
- if H rejects then reject.
- if H accepts, then simulate M on w until it halts.
- if at halting M has accepted w, accept, else reject.

But D decides L_{TM}^{A} which is undecidable. A contradiction.

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This proof strategy is called a reduction.

Reduction from the acceptance problem

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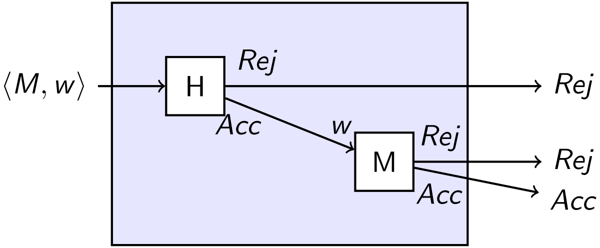

Some more undecidable problems

The emptiness problem for TMs

Does a given Turing machine accept any word?

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  - If R accepts, *reject*; if R rejects then *accept*.

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  - If R accepts, *accept*; if R rejects, *reject*.

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- note that every instance of P_2 need not be covered!

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Proof: Exercise.