CS310 : Automata Theory 2019

Lecture 29: Reductions contd.

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Pop-Quiz

Which is easier: Emptiness or non-emptiness?

- $L_e = \{\langle M \rangle \mid L(M) = \emptyset\}$
- $L_{ne} = \{\langle M \rangle \mid L(M) \neq \emptyset\}$

Recap

Turing machines and computability

1. Definition of Turing machines: high level and low-level descriptions
2. Variants of Turing machines
3. Decidable and Turing recognizable languages

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   - A universal TM lang $L^A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

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Regular $\subsetneq$ Decidable $\subsetneq$ Recursively Enumerable $\subsetneq$ All languages

DFA/NFA $<$ Algorithms/Halting TM $<$ Semi-algorithms/TM

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7. Today: Reductions and moar undecidability!

The principle of reduction

(Many-to-one) Reduction

- An algorithm (halting TM!) to convert instances of a problem P_1 to another P_2 such that,
  - answer is yes for P_1 iff answer is yes for P_2
  - answer is no for P_1 iff answer is no for P_2

Note that every instance of P_2 need not be covered!

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- if P_1 is undecidable, then so is P_2
- if P_1 is not r.e., then so is P_2.

The halting problem

The halting problem for Turing Machines is undecidable

Does a given Turing machine halt on a given input?

- $L_{TM}^{HALT} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$.

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Proof: Suppose there exists TM H deciding L_{TM}^{HALT}, then construct a TM D s.t., on input $\langle M, w\rangle$:

- runs TM H on input $\langle M, w\rangle$
- if H rejects then reject.
- if H accepts, then simulate M on w until it halts.
- if at halting M has accepted w, accept, else reject.

But D decides L_{TM}^{A} which is undecidable. A contradiction.

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This proof strategy is called a reduction.

Reduction from the acceptance problem

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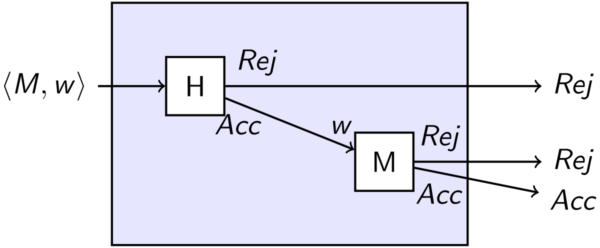

Some more undecidable problems

The emptiness problem for TMs

Does a given Turing machine accept any word?

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  - If R accepts,

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  - If R accepts, *reject*; if R rejects then *accept*.

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Proof: Reduce to emptiness.

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Rice's theorem (1953)

Any non-trivial property of R.E languages is undecidable!

- Property $P \equiv$ set of languages (i.e., their TM encodings) satisfying P
- Property of r.e languages: membership of M in P depends only on the language of M. If $L(M) = L(M')$, then $\langle M \rangle \in P$ iff $\langle M' \rangle \in P$.
- Non-trivial: It holds for some but not all TMs.