CS310 : Automata Theory 2019

Lecture 30: Rice theorem

Instructor: S. Akshay

IITB, India

19-03-2019

Recap

Turing machines and computability

1. Definition of Turing machines: high level and low-level descriptions
2. Variants of Turing machines
3. Decidable and Turing recognizable languages
4. Church-Turing Hypothesis

Recap

Turing machines and computability

1. Definition of Turing machines: high level and low-level descriptions
2. Variants of Turing machines
3. Decidable and Turing recognizable languages
4. Church-Turing Hypothesis
5. Undecidability and a proof technique by diagonalization
   - A universal TM lang $L_{TM}^{A} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
6. Reductions: a powerful way to show undecidability.

Recap

Turing machines and computability

1. Definition of Turing machines: high level and low-level descriptions
2. Variants of Turing machines
3. Decidable and Turing recognizable languages
4. Church-Turing Hypothesis
5. Undecidability and a proof technique by diagonalization
   - A universal TM lang $L^A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
6. Reductions: a powerful way to show undecidability.

$$\text{Regular} \subsetneq \text{Decidable} \subsetneq \text{Recursively Enumerable} \subsetneq \text{All languages}$$

$$\text{DFA/NFA} < \text{Algorithms/Halting TM} < \text{Semi-algorithms/TM}$$

Recap

Turing machines and computability

1. Definition of Turing machines: high level and low-level descriptions
2. Variants of Turing machines
3. Decidable and Turing recognizable languages
4. Church-Turing Hypothesis
5. Undecidability and a proof technique by diagonalization
   - A universal TM lang $L^A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
6. Reductions: a powerful way to show undecidability.
7. Today: Rice's theorem

Regular $\subsetneq$ Decidable $\subsetneq$ Recursively Enumerable $\subsetneq$ All languages

DFA/NFA $<$ Algorithms/Halting TM $<$ Semi-algorithms/TM

Rice's theorem

Rice's theorem (1953)

Any non-trivial property of R.E languages is undecidable!

Rice's theorem

Rice's theorem (1953)

Any non-trivial property of R.E languages is undecidable!

- Property $P \equiv$ set of languages satisfying P

Rice's theorem

Rice's theorem (1953)

Any non-trivial property of R.E languages is undecidable!

- Property $P \equiv$ set of languages satisfying P
- Property of r.e languages: membership of M in P depends only on the language of M. If $L(M) = L(M')$, then $\langle M \rangle \in P$ iff $\langle M' \rangle \in P$.

Rice's theorem

Rice's theorem (1953)

Any non-trivial property of R.E languages is undecidable!

- Property $P \equiv$ set of languages satisfying P
- Property of r.e languages: membership of M in P depends only on the language of M. If $L(M) = L(M')$, then $\langle M \rangle \in P$ iff $\langle M' \rangle \in P$.
- Non-trivial: It holds for some but not all TMs.

Property: set of languages

Property of r.e. languages

Just a subset of r.e. languages. Thus, L satisfies a property P iff $L \in P$.

Property: set of languages

Property of r.e. languages

Just a subset of r.e. languages. Thus, L satisfies a property P iff $L \in P$.

Examples:

- Set of regular languages
- Set of context-free languages

Property: set of languages

Property of r.e. languages

Just a subset of r.e. languages. Thus, L satisfies a property P iff $L \in P$.

Examples:

- Set of regular languages
- Set of context-free languages
- Set of all languages

Property: set of languages

Property of r.e. languages

Just a subset of r.e. languages. Thus, L satisfies a property P iff $L \in P$.

Examples:

- Set of regular languages
- Set of context-free languages
- Set of all languages
- $\{\emptyset\}$
- $\emptyset$

Property: set of languages

Property of r.e. languages

Just a subset of r.e. languages. Thus, L satisfies a property P iff $L \in P$.

Examples:

- Set of regular languages
- Set of context-free languages
- Set of all languages
- $\{\emptyset\}$
- $\emptyset$

Given a property P, denote $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$

So decidability of property P is decidability of language $\mathcal{L}_P$.

Non-trivial property

For a property P, $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$.

A property P of r.e. languages is called *non-trivial* if

- $\mathcal{L}_P \neq \emptyset$, i.e., there exists TM M, $L(M) \in P$.
- $\mathcal{L}_P \neq$ all r.e. languages, i.e., there exists TM M, $L(M) \notin P$.

Non-trivial property

For a property P, $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$.

A property P of r.e. languages is called *non-trivial* if

- $\mathcal{L}_P \neq \emptyset$, i.e., there exists TM M, $L(M) \in P$.
- $\mathcal{L}_P \neq$ all r.e. languages, i.e., there exists TM M, $L(M) \notin P$.

Examples

- $\{\langle M \rangle \mid L(M)$ is regular $\}$
- $\mathcal{L}_P = \{\langle M \rangle \mid L(M) = \emptyset\}$
- $\mathcal{L}_p = \{\langle M \rangle \mid M$ is a TM s.t. $L(M)$ is accepted by a TM with even number of states $\}$.

Rice's Theorem

Rice's Theorem

Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

Rice's Theorem

Rice's Theorem

Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

Remark: (when not to apply Rice's theorem)

1. Property is is about TMs and not their languages: e.g., TM has at least 10 states.

Rice's Theorem

Rice's Theorem

Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

Remark: (when not to apply Rice's theorem)

1. Property is is about TMs and not their languages: e.g., TM has at least 10 states.
2. Cannot show non- Turing recognizability, only undecidability.

Can you apply Rice's theorem?

For the following, answer if Rice's theorem is applicable

1. $\{\langle M \rangle \mid M$ runs for 5 steps on word 010$\}$.
2. $\{\langle M \rangle \mid L(M)$ is recognized by a TM with at least 25 states.$\}$.
3. $\{\langle M \rangle \mid L(M)$ is recognized by a TM with at most 25 states.$\}$.
4. $\{\langle M \rangle \mid M$ has at most 25 states.$\}$.
5. $\{\langle M \rangle \mid L(M)$ is finite.$\}$.
6. $\{\langle M \rangle \mid M$ with alphabet $\{0, 1, \sqcup\}$ ever prints three consecutive $1's$ on the tape$\}$.

Can you apply Rice's theorem?

For the following, answer if Rice's theorem is applicable

1. $\{\langle M \rangle \mid M$ runs for 5 steps on word 010$\}$. No. Property of TMs.
2. $\{\langle M \rangle \mid L(M)$ is recognized by a TM with at least 25 states.$\}$. No. Trivial property.
3. $\{\langle M \rangle \mid L(M)$ is recognized by a TM with at most 25 states.$\}$. Yes, if tape alphabet is fixed, else no!
4. $\{\langle M \rangle \mid M$ has at most 25 states.$\}$. No. Property of TMs.
5. $\{\langle M \rangle \mid L(M)$ is finite.$\}$. Yes.
6. $\{\langle M \rangle \mid M$ with alphabet $\{0, 1, \sqcup\}$ ever prints three consecutive $1's$ on the tape$\}$. No. Property of TMs.

- For each of No answers above, is the language decidable?
- What do you do when Rice's theorem does not apply? Fall back on reductions!

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P
- We combine M_L and M_P to get algo for A_{TM}.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P
- We combine M_L and M_P to get algo for A_{TM}.
  - Take $\langle M, w \rangle$ as i/p, and o/p $\langle M' \rangle$, s.t $L(M') \in P$ iff M acc w.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then
$\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P
- We combine M_L and M_P to get algo for A_{TM}.
  - Take $\langle M, w \rangle$ as i/p, and o/p $\langle M' \rangle$, s.t $L(M') \in P$ iff M acc w.
  - M' is the foll:
    1. ignore i/p x , simulate M on w. if reject, then rejects x.
    2. if acc, then simulate M_L on x, acc iff M_L acc x.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then $\mathcal{L}_P = \{\langle M\rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P
- We combine M_L and M_P to get algo for A_{TM}.
  - Take $\langle M, w\rangle$ as i/p, and o/p $\langle M'\rangle$, s.t $L(M') \in P$ iff M acc w.
  - M' is the foll:
    1. ignore i/p x , simulate M on w. if reject, then rejects x.
    2. if acc, then simulate M_L on x, acc iff M_L acc x.
  - Thus M' either acc $\emptyset$ or L depending on if M acc w.

Proof idea

Rice's Theorem
Let P be a non-trivial property of r.e. languages. Then
$\mathcal{L}_P = \{\langle M \rangle \mid L(M) \in P\}$ is undecidable.

- Let P be a non-trivial property of r.e, such that $\emptyset \notin P$.
- Assume $\mathcal{L}_P$ is decidable, we will show that A_{TM} is decidable.
- Since P is non-trivial, $\exists L, M_L$ with property P.
- If P is decidable, there exists an algo M_P for deciding P
- We combine M_L and M_P to get algo for A_{TM}.
  - Take $\langle M, w \rangle$ as i/p, and o/p $\langle M' \rangle$, s.t $L(M') \in P$ iff M acc w.
  - M' is the foll:
    1. ignore i/p x , simulate M on w. if reject, then rejects x.
    2. if acc, then simulate M_L on x, acc iff M_L acc x.
  - Thus M' either acc $\emptyset$ or L depending on if M acc w.
- This gives an algo for A_{TM} so completes the proof.

Proof idea

- But what if P has $\emptyset$ in it!?

Proof idea

- But what if P has $\emptyset$ in it!?
- Take $\overline{P}$.
- Now $\emptyset \notin \overline{P}$.

Proof idea

- But what if P has $\emptyset$ in it!?
- Take $\overline{P}$.
- Now $\emptyset \notin \overline{P}$.
- Apply proof to get undecidability of $\mathcal{L}_{\overline{P}}$.
- Conclude undecidability of $\mathcal{L}_P$.