CS310 : Automata Theory 2019

Lecture 40: Efficiency in computation
Classifying problems by their complexity

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Recap

Turing machines and computability

1. Turing machines
   (i) Definition & variants
   (ii) Decidable and Turing recognizable languages
   (iii) Church-Turing Hypothesis

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   (i) A proof technique by diagonalization
   (ii) Via reductions
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3. Applications: showing (un)decidability of other problems
   (i) A string matching problem: Post's Correspondance Problem
   (ii) A problem for compilers: Unambiguity of Context-free languages
   (iii) Between TM and PDA: Linear Bounded Automata

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3. Applications: showing (un)decidability of other problems
   (i) A string matching problem: Post's Correspondance Problem
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   (iii) Between TM and PDA: Linear Bounded Automata
4. Efficiency in computation: run-time complexity.
   (i) Running time complexity: polynomial and exponential time
   (ii) Nondeterministic polynomial time, and the P vs NP problem.
   (iii) NP-completeness, the Cook-Levin Theorem.

The P vs NP problem

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  $=$ class of problems solvable in polynomial time in a non-determistic TM.
- EXP: class of problems solvable in exponential time.
- $NEXP$: class of problems solvable in exponential time by a non-det TM.
- $Co-\mathcal{C}$: class of problems whose complement is solvable in $\mathcal{C}$.

Polynomial time reductions

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Language A polynomial time reducible to B, denoted $A \leq_P B$ if there is a Ptime computable function f s.t.

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(Note: if there is a "halting TM" reduction from A to B, then A undecidable implied B undecidable!)

Exercise (H.W)

- Show that $3SAT$ is polytime reducible to CLIQUE.
- Show that $3SAT$ is polytime reducible to SUBSETSUM.

NP-completeness

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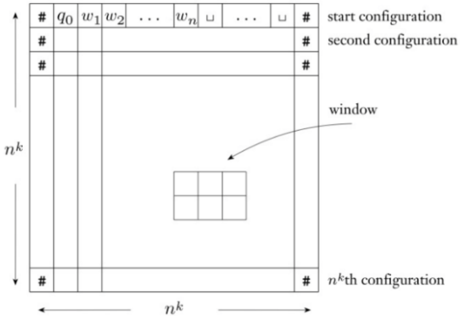
4. Thus, N acc w iff there exists an accepting tableau for N on w.

Cook-Levin Theorem (Contd.)



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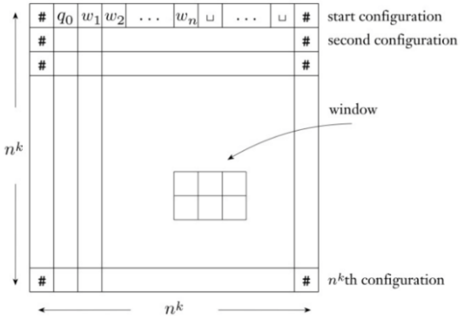

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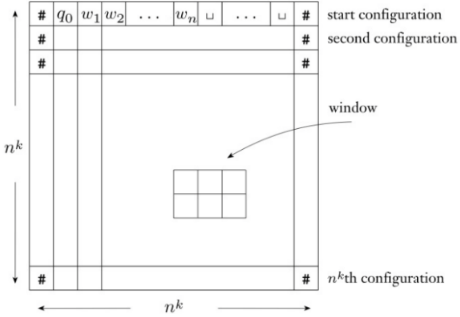

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1. Variable $x_{i,j,s}$ for each $1 \leq i, j \leq n^k$, $s \in C = Q \cup \Gamma \cup \{\#\}$.

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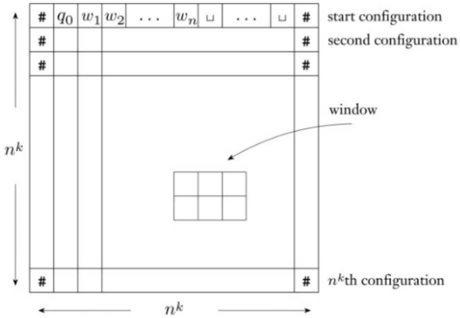

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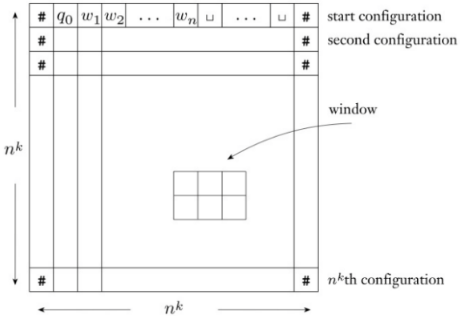

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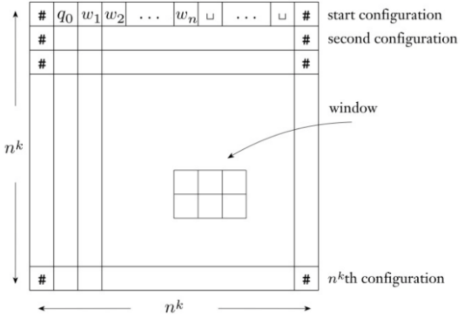

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4. $\varphi = \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{move} \wedge \varphi_{acc}$

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4. move encodes that each row correponds to config that "legally" follows the preceding row config acc to N
   $\varphi_{move} = \bigwedge_{1, \leq i,j < n^k}$ (the (i,j)-window is legal)