CS310 : Automata Theory 2019

Lecture 41: Efficiency in computation
Classifying problems by their complexity

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Recap

Turing machines and computability

1. Turing machines
   (i) Definition & variants
   (ii) Decidable and Turing recognizable languages
   (iii) Church-Turing Hypothesis

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3. Applications: showing (un)decidability of other problems
   (i) A string matching problem: Post's Correspondance Problem
   (ii) A problem for compilers: Unambiguity of Context-free languages
   (iii) Between TM and PDA: Linear Bounded Automata

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3. Applications: showing (un)decidability of other problems
   (i) A string matching problem: Post's Correspondance Problem
   (ii) A problem for compilers: Unambiguity of Context-free languages
   (iii) Between TM and PDA: Linear Bounded Automata
4. Efficiency in computation: run-time complexity.
   (i) Running time complexity: polynomial and exponential time
   (ii) Nondeterministic polynomial time, and the P vs NP problem.
   (iii) NP-completeness, the Cook-Levin Theorem.

NP-completeness

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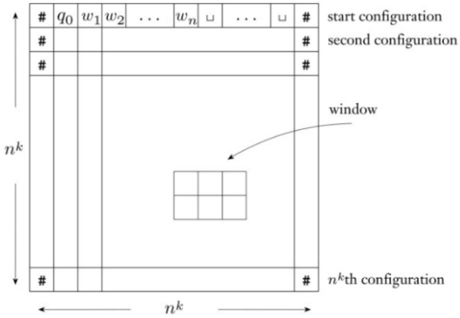
4. Thus, N acc w iff there exists an accepting tableau for N on w.

Cook-Levin Theorem (Contd.)



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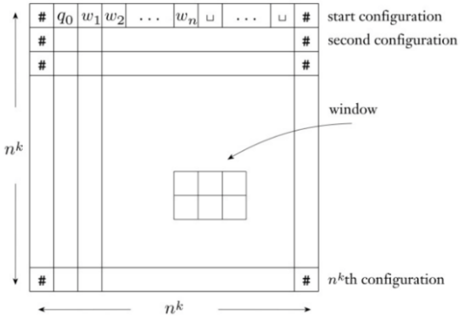

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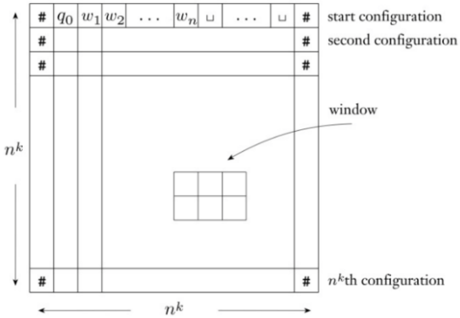

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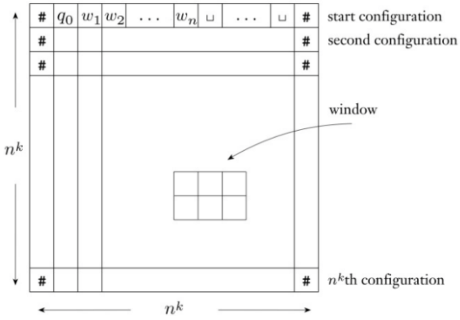

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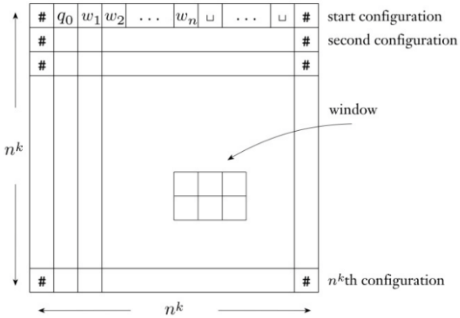

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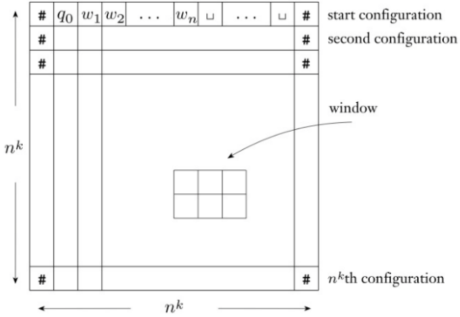

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4. $\varphi = \varphi_{cell} \wedge \varphi_{start} \wedge \varphi_{move} \wedge \varphi_{acc}$

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   $\varphi_{move} = \bigwedge_{1, \leq i,j < n^k}$ (the (i,j)-window is legal)

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3. Ex: Give 3 examples of legal windows and 3 illegal windows, given $\delta(q, a) = \{q, b, R\}$, $\delta(q, b) = \{(q', c, L), (q', a, R)\}$

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Encode this as a disjunct $\bigvee_{a_1 \ldots a_6 \text{is legal}} (x_{i,j-1,a_1}) \wedge x_{i,j,a_2} \wedge \ldots)$

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Thus, we have a PTime reduction from any NP problem to SAT, i.e., SAT is NP-hard.

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Well, there are some natural problems that are not known to be in P, but that are in NP and not known to be NP-hard.

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An algorithmician's answer:

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3. Goal nowadays: develop efficient encoding into SAT!