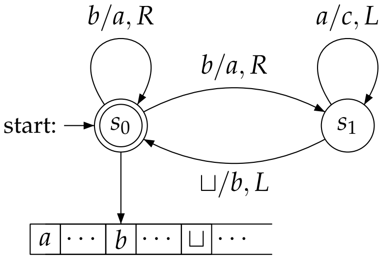
CS 208: Automata Theory and Logic

Part II, Lecture 3: Reductions

S Akshay



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Summary of previous lecture

$$\text{Regular} \subsetneq \text{Decidable} \subsetneq \text{Recursively Enumerable} \subsetneq \text{All languages}$$

$$\text{DFA/NFA} < \text{Algorithms/Halting TM} < \text{Semi-algorithms/TM}$$

Properties

1. There exist languages that are not R.E.
2. There exist languages that are R.E but are undecidable.
   Eg. universal TM lang $L_{TM}^{A} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$
3. Decidable languages are closed under complementation.
4. L is decidable iff L is $R.E$ and $\overline{L}$ is also R.E.

The halting problem

The halting problem for Turing Machines is undecidable

Does a given Turing machine halt on a given input?

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Proof: Suppose there exists TM H deciding L_{TM}^{HALT}, then construct a TM D s.t., on input $\langle M, w\rangle$:

- runs TM H on input $\langle M, w\rangle$
- if H rejects then reject.
- if H accepts, then simulate M on w until it halts.
- if at halting M has accepted w, accept, else reject.

But D decides L_{TM}^{A} which is undecidable. A contradiction.

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This proof strategy is called a reduction.

Reduction from the acceptance problem

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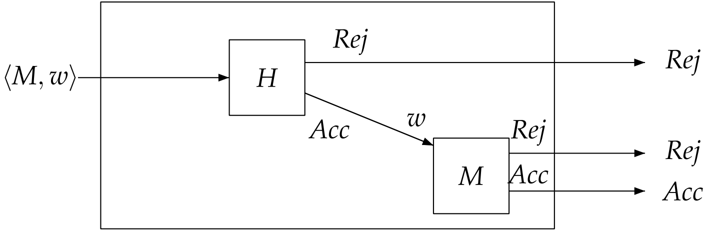

Some more undecidable problems

The emptiness problem for TMs

Does a given Turing machine accept any word?

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The regularity problem for TMs

Does a given Turing machine accept a regular language?

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Rice's Theorem

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Rice's theorem (1953)

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- Property $P \equiv$ set of languages (i.e., their TM encodings) satisfying P
- Property of r.e languages: membership of M in P depends only on the language of M. If $L(M) = L(M')$, then $\langle M \rangle \in P$ iff $\langle M' \rangle \in P$.
- Non-trivial: It holds for some but not all TMs.

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