Lagrange Interpolation and Neville's Algorithm

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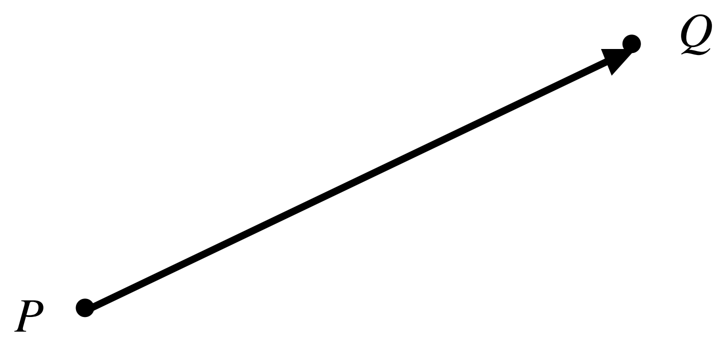
Tension between Mathematics and Engineering

1. How do Mathematicians actually represent curves and surfaces?
   - Algebra -- Formulas and Algorithms

2. How do Scientists and Engineers want to represent curves and surfaces?
   - Geometry -- Interpolation and Approximation

Straight Lines



Equation = ?

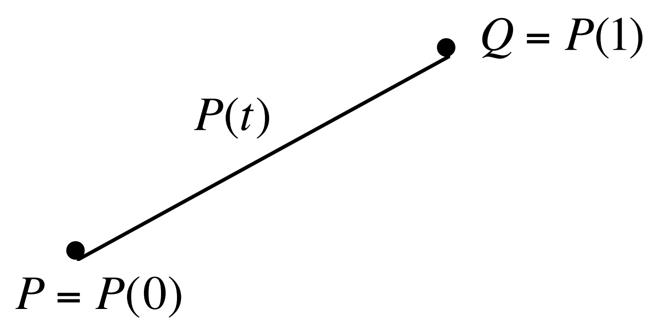
<u>**Linear Interpolation**</u>

Straight Line

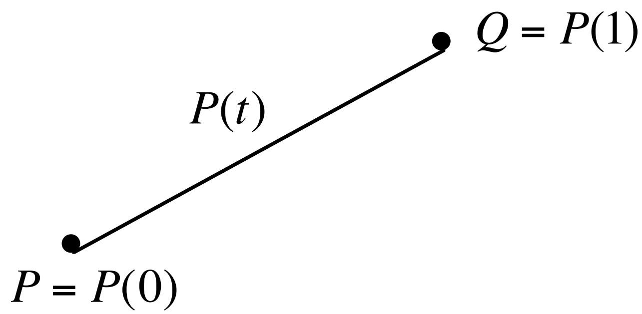
- $P(t) = P + t(Q - P)$
- $P(t) = (1 - t)P + tQ$

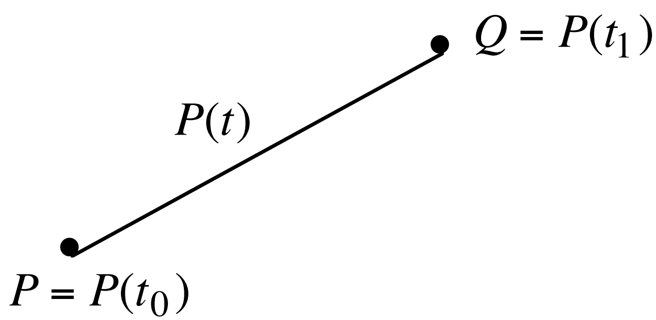
Observations

- $P(0) = P$
- $P(1) = Q$



Linear Interpolation

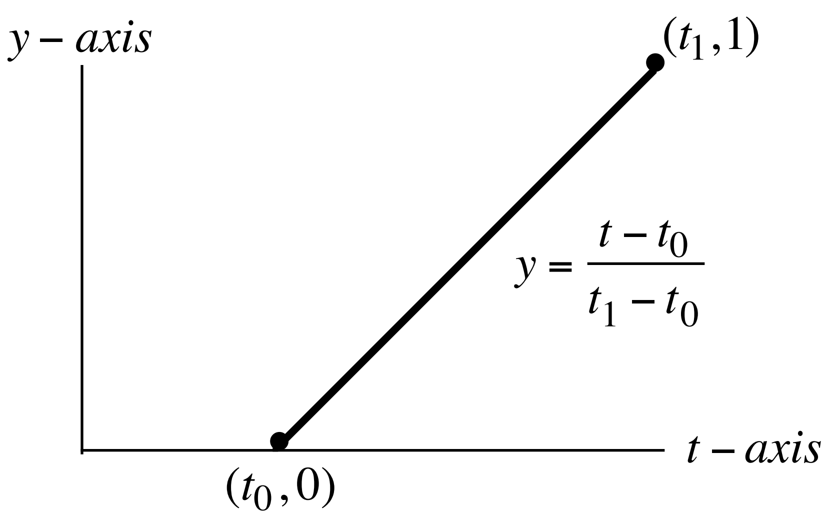




Linear Interpolation Revisited

Straight Line

- $P_{01}(t) = (1 - f(t))P_0 + f(t)P_1$
- $f(t_0) = 0$ and $f(t_1) = 1$



<u>**Linear Interpolation Revisited**</u>

Linear Interpolation

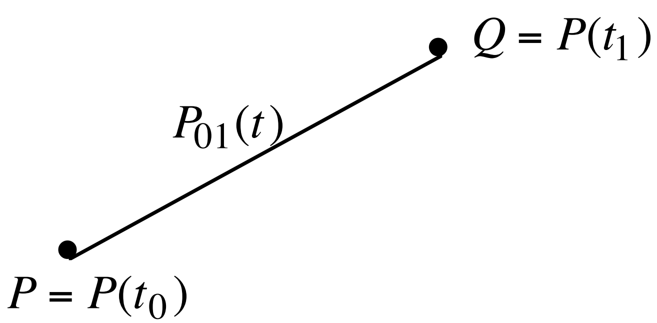
- $f(t) = \dfrac{(t - t_0)}{(t_1 - t_0)}$

- $f(t_0) = 0$ and $\quad f(t_1) = 1.$

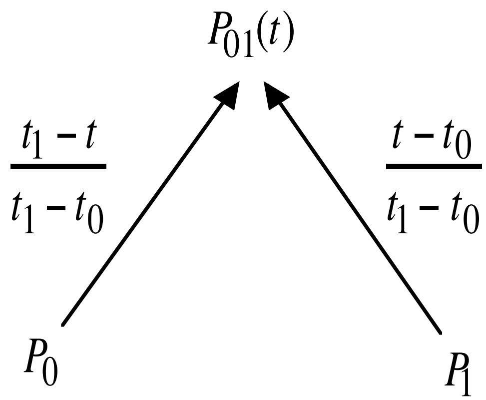
Straight Line

- $P_{01}(t) = \dfrac{t_1 - t}{t_1 - t_0} P_0 + \dfrac{t - t_0}{t_1 - t_0} P_1$

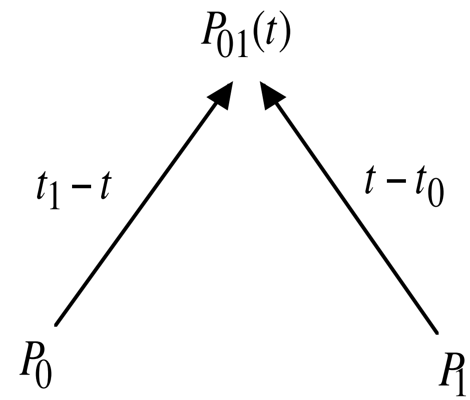
- $P(t_0) = P \qquad P(t_1) = Q$



Linear Interpolation



Normalized

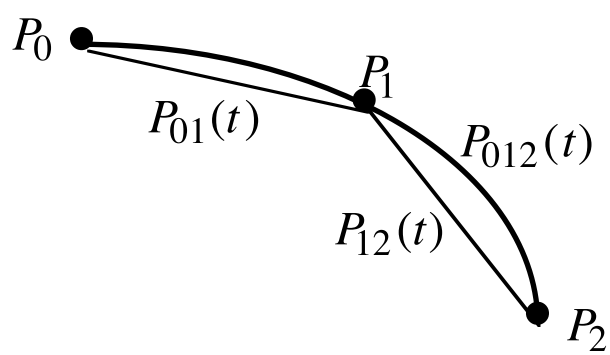


Unnormalized

$$P_{01}(t) = \frac{t_1 - t}{t_1 - t_0} P_0 + \frac{t - t_0}{t_1 - t_0} P_1$$

$$P_{01}(t_0) = P_0 \qquad\qquad P_{01}(t_1) = P_1$$

Quadratic Interpolation



Problem

Find a smooth curve $P_{012}(t)$ such that:

$P_{012}(t_0) = P_0$ $\qquad$ $P_{012}(t_1) = P_1$ $\qquad$ $P_{012}(t_2) = P_2$

Quadratic Interpolation

Linear Interpolation

- $$P_{01}(t) = \frac{t_1 - t}{t_1 - t_0} P_0 + \frac{t - t_0}{t_1 - t_0} P_1$$

- $$P_{12}(t) = \frac{t_2 - t}{t_2 - t_1} P_1 + \frac{t - t_1}{t_2 - t_1} P_2$$

Quadratic Interpolation

- $$P_{012}(t) = \frac{t_2 - t}{t_2 - t_0} P_{01}(t) + \frac{t - t_0}{t_2 - t_0} P_{12}(t)$$

Verification of Quadratic Interpolation

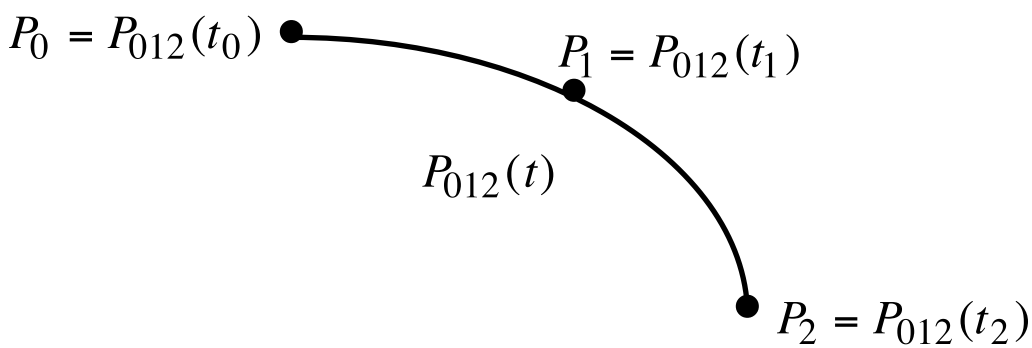
$$P_{012}(t) = \frac{t_2 - t}{t_2 - t_0} P_{01}(t) + \frac{t - t_0}{t_2 - t_0} P_{12}(t)$$

i. $P_{012}(t_0) = P_{01}(t_0) = P_0$

ii. $P_{012}(t_2) = P_{12}(t_2) = P_2$

iii. $P_{012}(t_1) = \dfrac{t_2 - t_1}{t_2 - t_0} P_{01}(t_1) + \dfrac{t_1 - t_0}{t_2 - t_0} P_{12}(t_1)$

$$= \frac{t_2 - t_1}{t_2 - t_0} P_1 + \frac{t_1 - t_0}{t_2 - t_0} P_1 = P_1$$

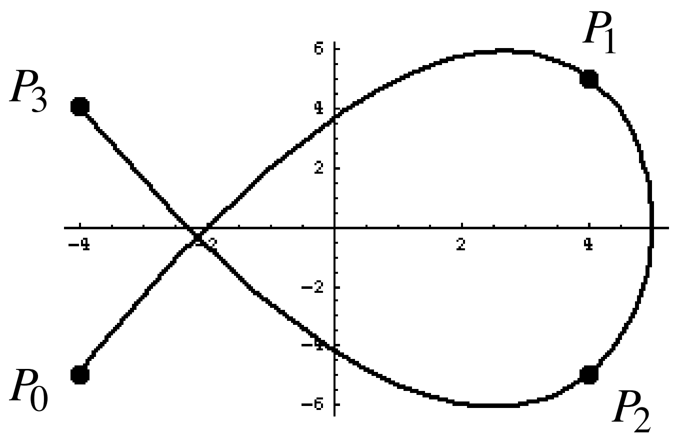


Neville's Algorithm for Quadratic Interpolation

$$
\begin{tikzcd}
 & & P_{012}(t) & & \\
 & P_{01}(t) \arrow[ur, "t_2 - t"] & & P_{12}(t) \arrow[ul, "t - t_0"'] & \\
P_0 \arrow[ur, "t_1 - t"] & & P_1 \arrow[ul, "t - t_0"'] \arrow[ur, "t_2 - t"] & & P_2 \arrow[ul, "t - t_1"']
\end{tikzcd}
$$

$$P_{012}(t_k) = P_k$$

Cubic Interpolation

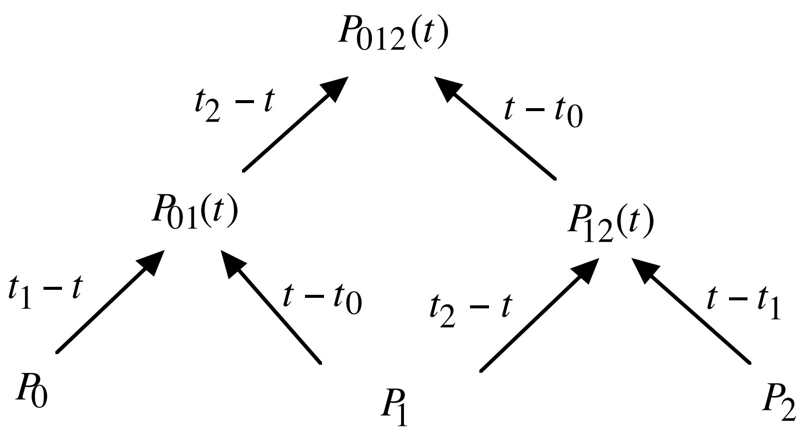


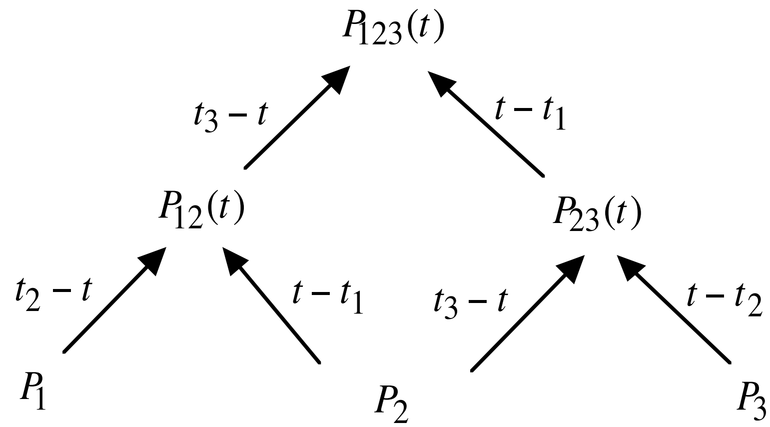
Problem

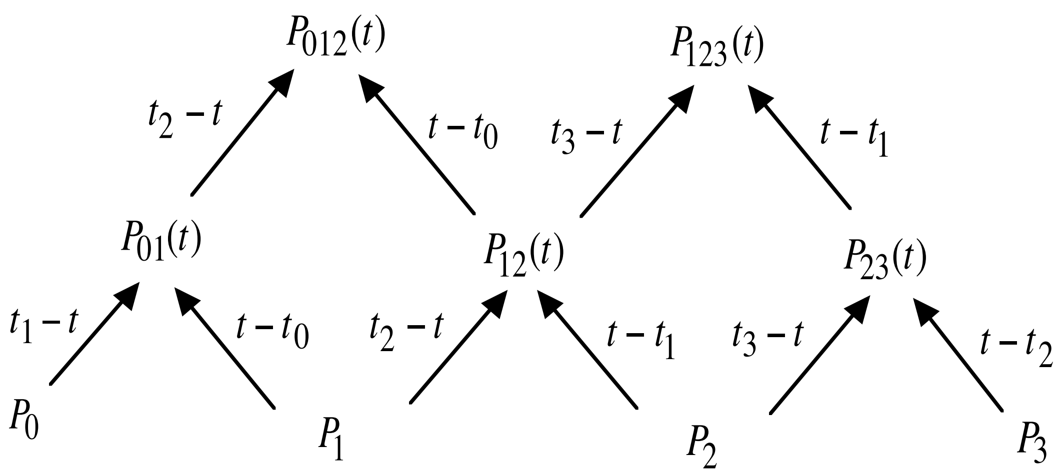
Find a smooth function $P_{0123}(t)$ such that:

$P_{0123}(t_0) = P_0 \qquad P_{0123}(t_1) = P_1 \qquad P_{0123}(t_2) = P_2 \qquad P_{0123}(t_3) = P_3$

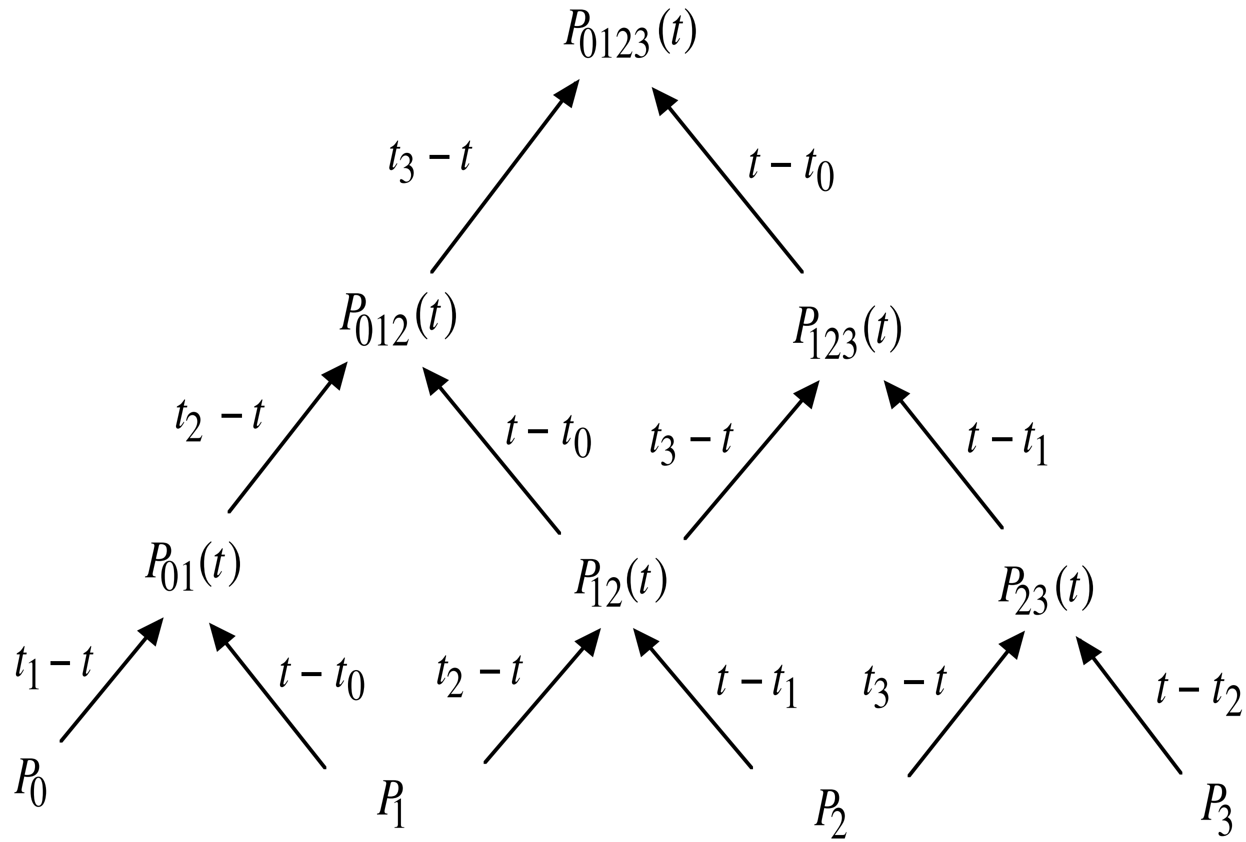
Neville's Algorithm for Two Quadratic Curves







Neville's Algorithm for Cubic Curves



$$P_{0123}(t_k) = P_k$$

Verification of Cubic Interpolation

$$P_{0123}(t) = \frac{t_3 - t}{t_3 - t_0} P_{012}(t) + \frac{t - t_0}{t_3 - t_0} P_{123}(t)$$

i. $P_{0123}(t_0) = P_{012}(t_0) = P_0$

ii. $P_{0123}(t_3) = P_{123}(t_3) = P_3$

iii.
$$\begin{aligned} P_{0123}(t_1) &= \frac{t_3 - t_1}{t_3 - t_0} P_{012}(t_1) + \frac{t_1 - t_0}{t_3 - t_0} P_{123}(t_1) \\ &= \frac{t_3 - t_1}{t_3 - t_0} P_1 + \frac{t_1 - t_0}{t_3 - t_0} P_1 \\ &= P_1 \end{aligned}$$

iv. $P_{0123}(t_2) = P_2$ (same as proof for t_1)

Neville's Algorithm for Lagrange Interpolation

Theorem: *Given points* $P_0, ..., P_n$ *and parameters* $t_0, ..., t_n$, *there exists a polynomial curve* $P_{0\cdots n}(t)$ *of degree n that interpolates the given points at the specified parameter values. That is,*

$$P_{0\cdots n}(t_k) = P_k \qquad k = 0, ..., n \ .$$

Proof: By induction on n. Define

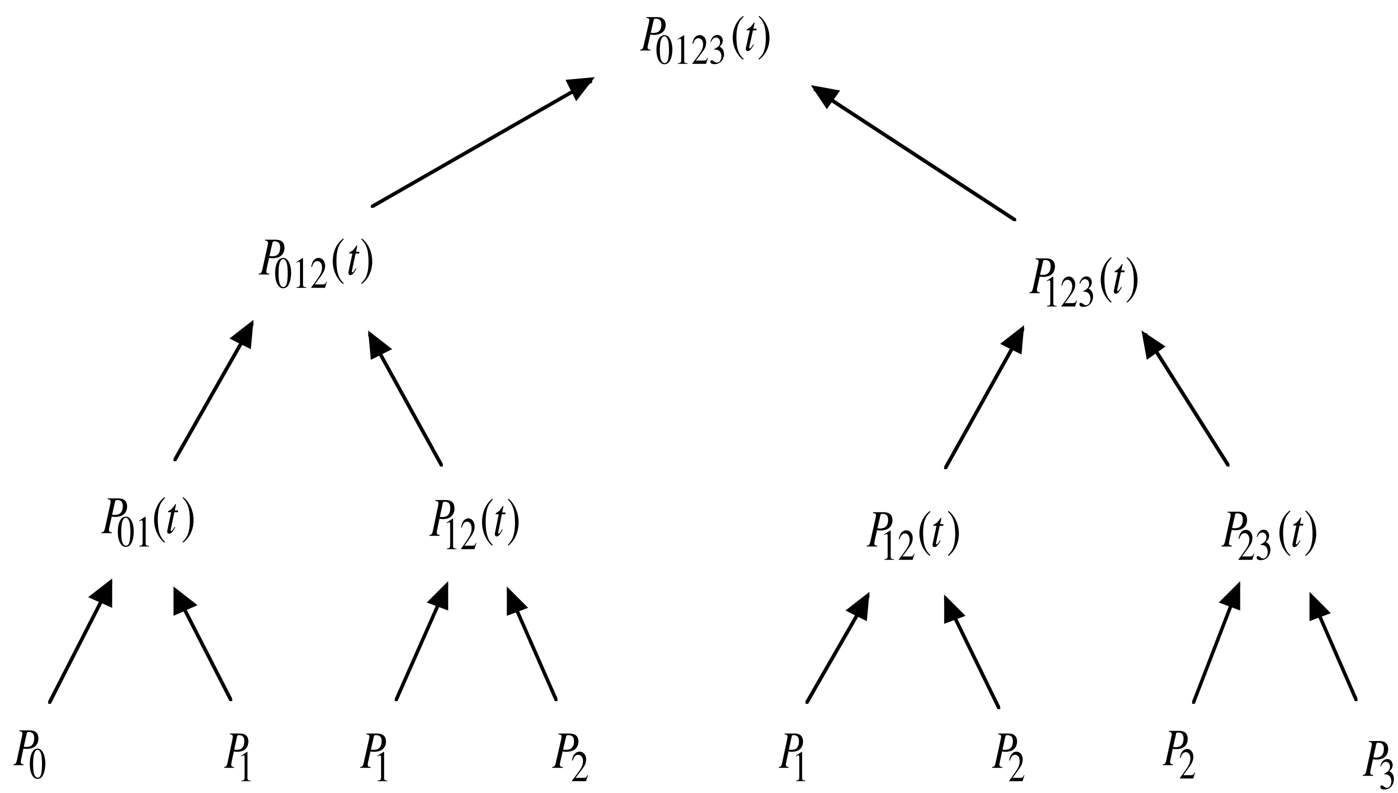
$$P_{0\cdots n}(t) = \frac{t_n - t}{t_n - t_0} P_{0\cdots n-1}(t) + \frac{t - t_0}{t_n - t_0} P_{1\cdots n}(t) \ .$$

Applying the same arguments we used in the quadratic and cubic cases, you can easily verify that

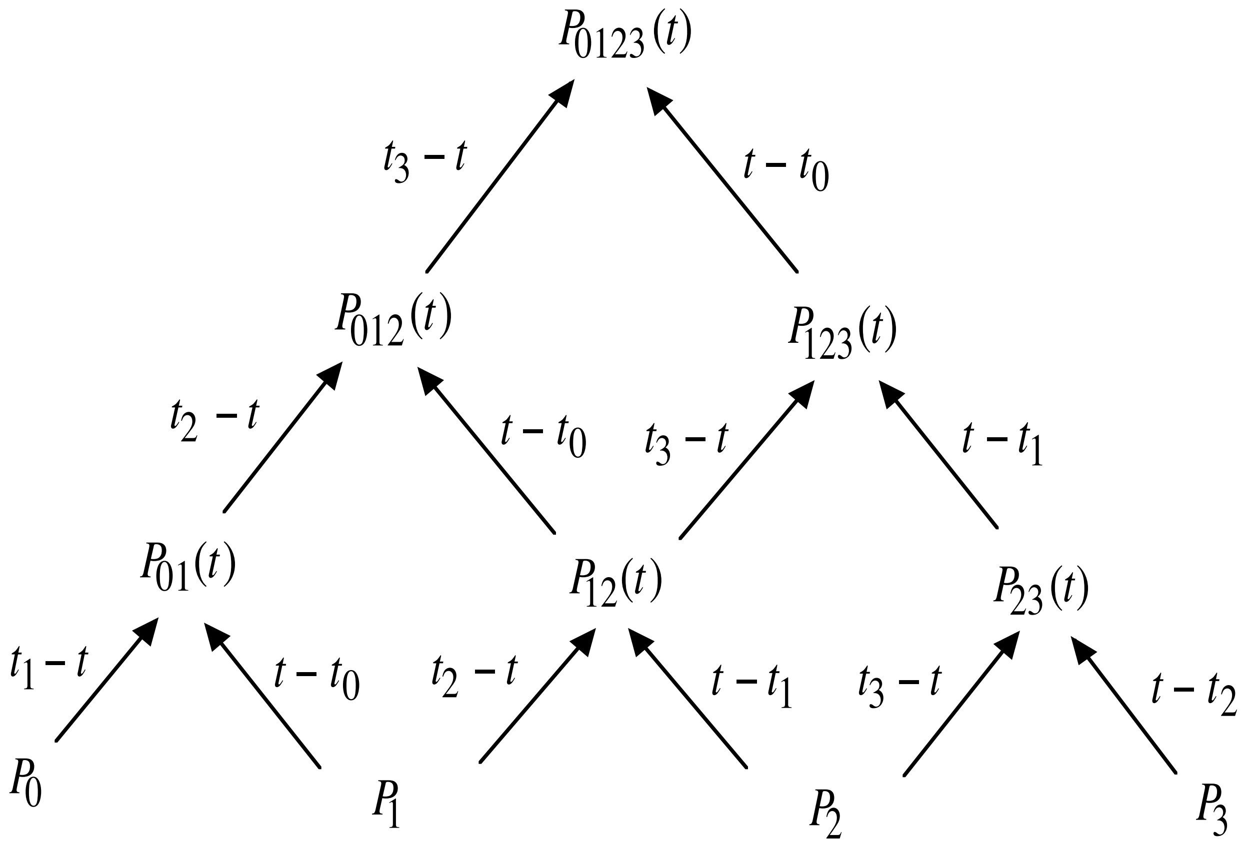
$$P_{0\cdots n}(t_k) = P_k \qquad\qquad k = 0, ..., n \ .$$

Since $P_{0\cdots n-1}(t)$ and $P_{1\cdots n}(t)$ are polynomials of degree $n-1$, it follows that $P_{0\cdots n}(t)$ is a polynomial of degree n. ♦

Neville's Algorithm -- Recursive Calls



Neville's Algorithm -- Dynamic Programming



$$P_{0123}(t_k) = P_k$$

Polynomial Algebra

Theorem: *A non-zero polynomial of degree less than or equal to n can have at most n roots.*

Proof: Recall that if $P(t)$ is a polynomial, then

$$r \text{ is a root of } P(t) \Leftrightarrow t - r \text{ is a factor of } P(t).$$

Now a polynomial of degree at most n can have at most n linear factors.

Therefore a polynomial of degree less than or equal to n can have at most n roots.

♦

Corollary: *The only polynomial of degree less than or equal to n with more than n roots is the zero polynomial.*

Uniqueness of Lagrange Interpolation

Theorem: *Given points $P_0,\ldots,P_n$ and parameters $t_0,\ldots,t_n$, there exists only one polynomial curve $P_{0\cdots n}(t)$ of degree n that interpolates the given points at the specified parameter values. That is, the curve generated by Neville's algorithm is unique*

Proof: Suppose that there are two polynomials of degree n such that

$$P_{0\cdots n}(t_k) = P_k \qquad k = 0,\ldots,n$$

$$Q_{0\cdots n}(t_k) = P_k \qquad k = 0,\ldots,n\,.$$

Define

$$R_{0\cdots n}(t) = Q_{0\cdots n}(t) - P_{0\cdots n}(t).$$

Then $R_{0\cdots n}(t)$ is a polynomial of degree at most n. But

$$R_{0\cdots n}(t_k) = Q_{0\cdots n}(t_k) - P_{0\cdots n}(t_k) = 0 \qquad k = 0,\ldots,n\,,$$

so $R_{0\cdots n}(t)$ has $n+1$ roots. Therefore $R_{0\cdots n}(t)$ must be the zero polynomial, so

$$P_{0\cdots n}(t) = Q_{0\cdots n}(t). \qquad \blacklozenge$$

Uniqueness of Lagrange Interpolation (continued)

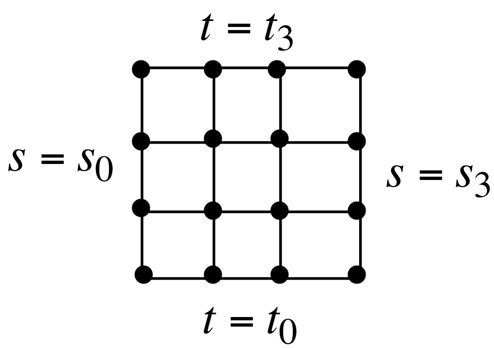
Corollary: *The Lagrange interpolant reproduces polynomials. That is, if the points* $P_0,\ldots,P_n$ *lie at the parameters* $t_0,\ldots,t_n$ *on a polynomial* $P(t)$ *of degree less than or equal to n, then* $P_{0\cdots n}(t)=P(t)$.

Observations

- Uniqueness applies only for fixed nodes.
- Changing the nodes $t_0,\ldots,t_n$, changes the Lagrange interpolant, even if the interpolation points $P_0,\ldots,P_n$ are exactly the same.

Tensor Product Surface Interpolation

Setup



(a) Domain -- Rectangular Grid

$$\begin{array}{cccc}
P_{03} & P_{13} & P_{23} & P_{33} \\
P_{02} & P_{12} & P_{22} & P_{32} \\
P_{01} & P_{11} & P_{21} & P_{31} \\
P_{00} & P_{10} & P_{20} & P_{30}
\end{array}$$

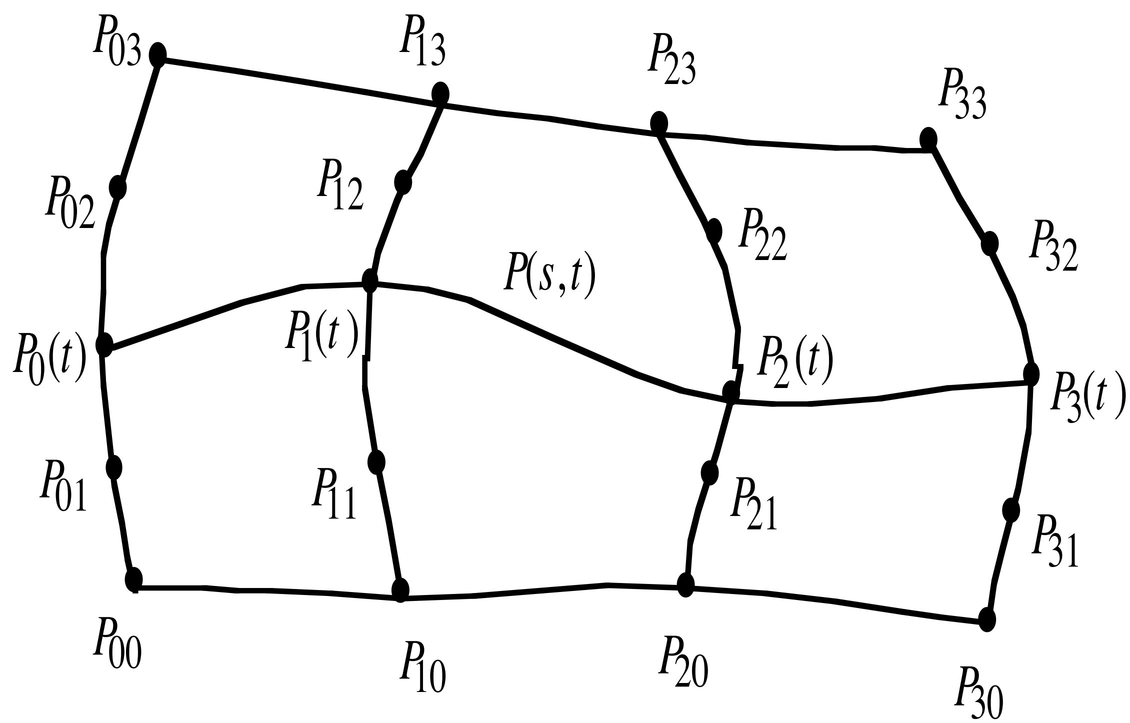
(b) Range -- Rectangular Array of Points

Problem

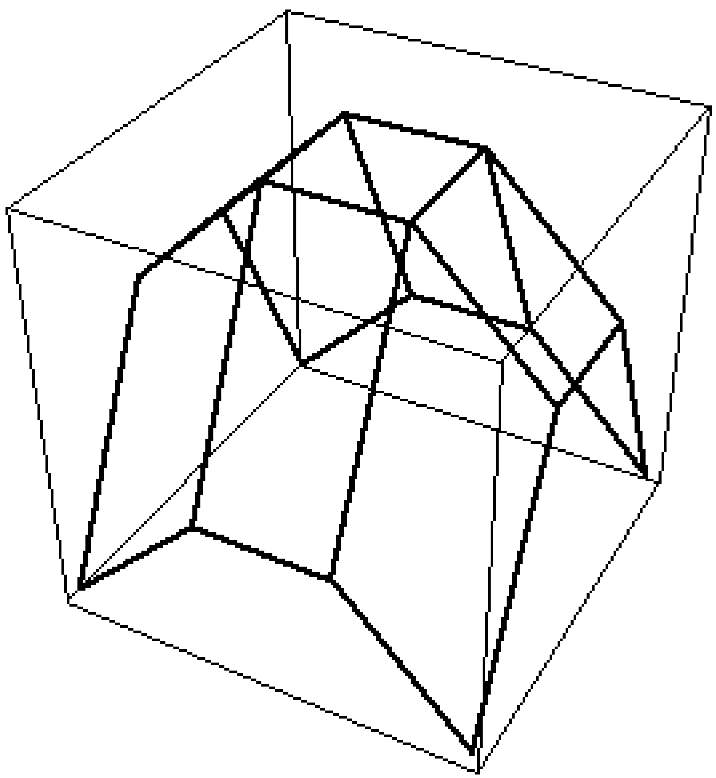
Find a <u>smooth</u> surface $P(s,t)$ such that:

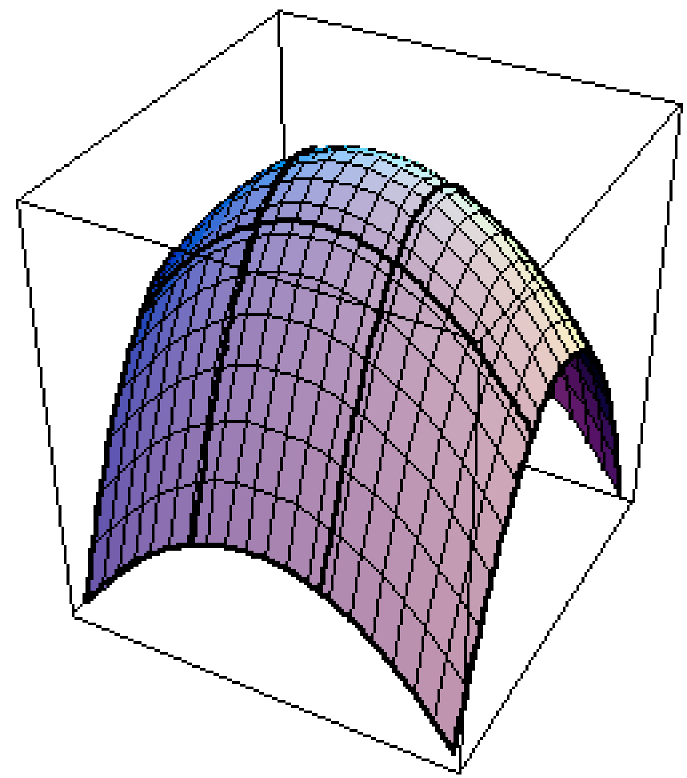
$$P(s_i, t_j) = P_{ij}$$

Surface Interpolation

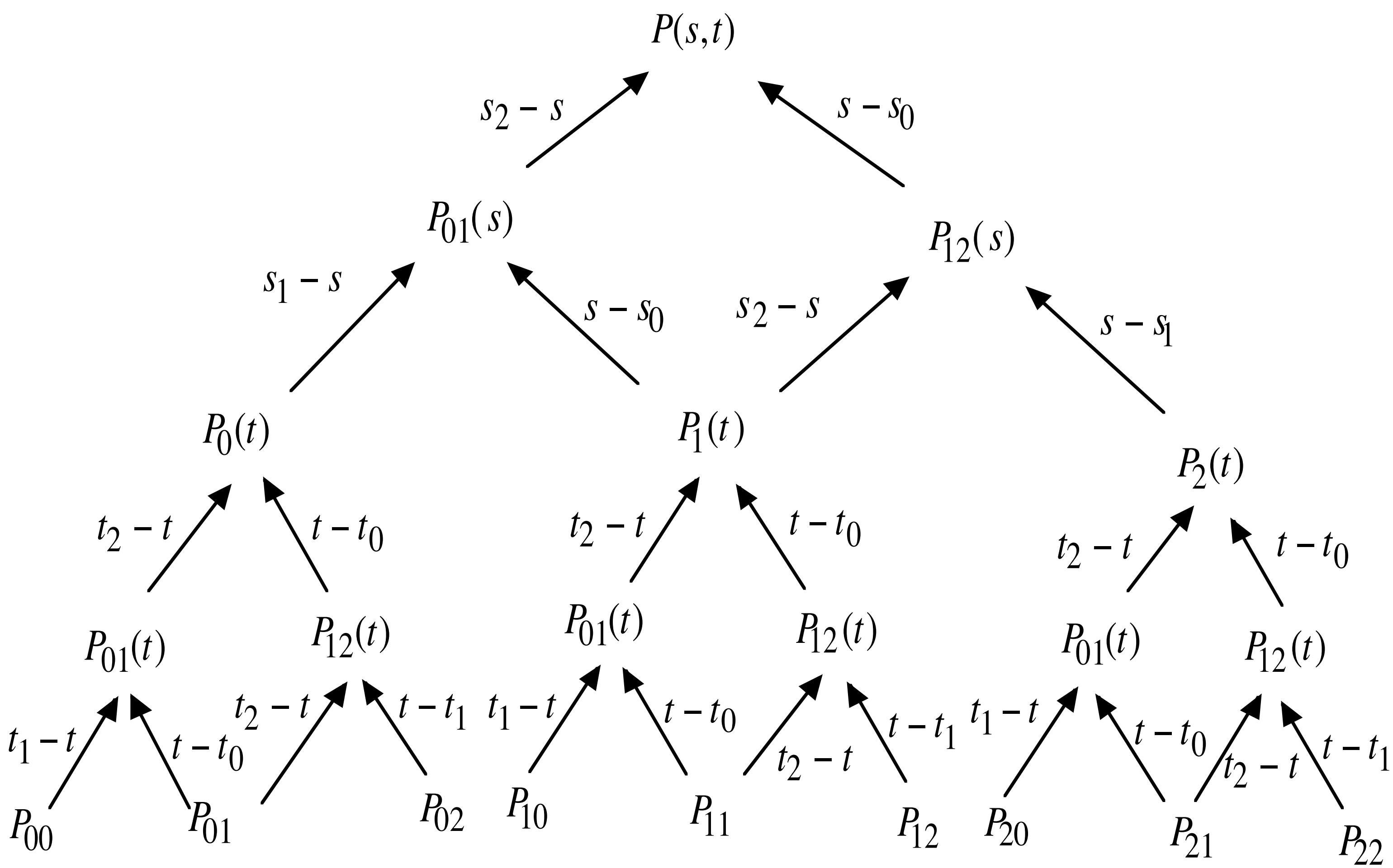


Surface Interpolation

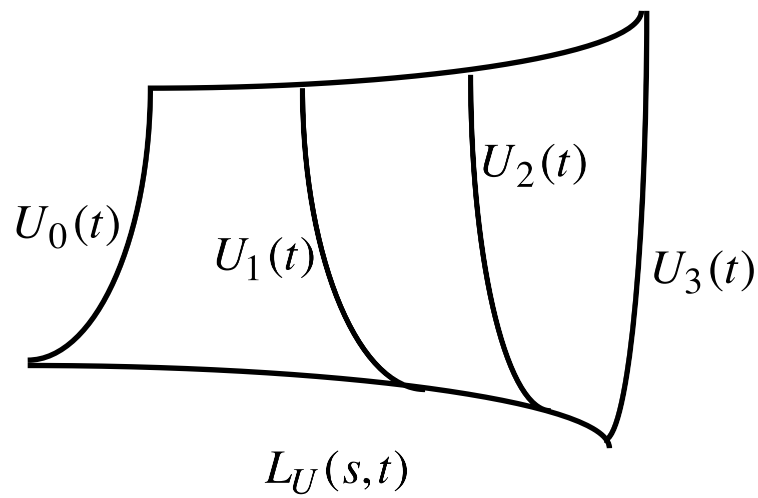




Neville's Algorithm for Tensor Product Surfaces



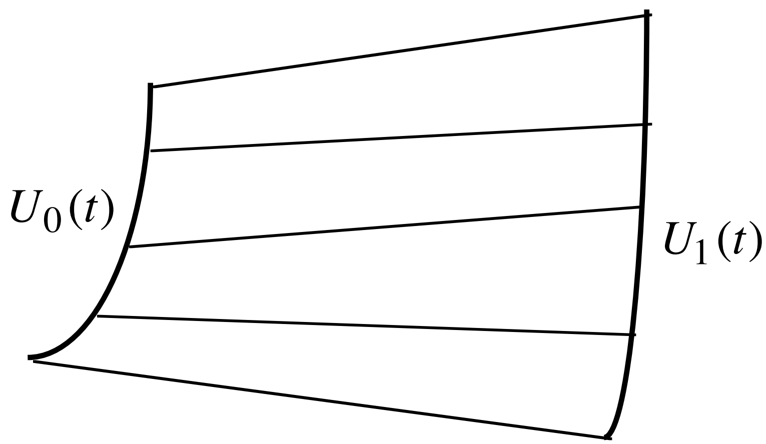
Lofted Surface



Neville's Algorithm for Lofted Surfaces

$$
\begin{tikzcd}
 & & & L_U(s,t) = U_{0123}(s,t) & & & \\
 & & U_{012}(s,t) \arrow[ur, "s_3 - s"] & & U_{123}(s,t) \arrow[ul, "s - s_0"'] & & \\
 & U_{01}(s,t) \arrow[ur, "s_2 - s"] & & U_{12}(s,t) \arrow[ul, "s - s_0"'] \arrow[ur, "s_3 - s"] & & U_{23}(s,t) \arrow[ul, "s - s_1"'] & \\
U_0(t) \arrow[ur, "s_1 - s"] & & U_1(t) \arrow[ul, "s - s_0"'] \arrow[ur, "s_2 - s"] & & U_2(t) \arrow[ul, "s - s_1"'] \arrow[ur, "s_3 - s"] & & U_3(t) \arrow[ul, "s - s_2"']
\end{tikzcd}
$$

Ruled Surface



$$R(s,t) = \frac{s_1 - s}{s_1 - s_0} U_0(t) + \frac{s - s_0}{s_1 - s_0} U_1(t)$$

Summary

Key Ideas

- Linear Interpolation
- Dynamic Programming -- Neville’s Algorithm
- Extensions to Surfaces
  - Tensor Product
  - Lofted
  - Ruled

Themes

Linearity

- Mathematics is Easy
- Represent Complicated Curves and Surfaces by (Successive) Linear Interpolations

Polynomial Curves and Surfaces

- Lagrange Interpolation -- Neville's Algorithm
- Bezier Approximation -- de Casteljau's Algorithm
- B-Splines -- de Boor's Algorithm
- Blossoming