30 years of

Zero Knowledge Proofs

Muthuramakrishnan Venkitasubramaniam

UNIVERSITY *of* ROCHESTER

An example

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EVENTS EUCLID

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

An example

S

I (re)solved “P vs NP?”

How?

Here is the proof π

π

Million $

Clay Institute

Can I convince someone the validity of something without revealing the proof?

Can I reveal “zero-knowledge” about a proof?

Proof Systems

Proof systems

$$L = \{(A, 1^k) : A \text{ is a true mathematical assertion of proof length } k\}$$

What is a "proof"?

Insight: meaningless unless can be **efficiently** verified

Proof systems

Given language L, goal is to prove $x \in L$

Proof system for L is a verification algorithm V

- Completeness: $\forall x \in L,\ \exists \Pi,\ V \text{ accepts } (x, \Pi)$

  "true assertions have proofs"

- Soundness: $\forall x \notin L,\ \forall \Pi^*,\ V \text{ rejects } (x, \Pi^*)$

  "false assertions have no proofs"

- Efficiency: V runs in polynomial time in |x|

Classical Proofs (a.k.a NP)

Previous definition: "classical" proof system

$L \in \mathsf{NP}$ iff expressible as

$$L = \{x \mid \exists y \text{ s.t. } |y| < |x|^k \text{ and } (x, y) \in R\}$$

where R is polynomial time computable

NP is the set of languages with classical proof systems

Interactive Proofs [GMR85]

Prover

Alice

Verifier

Bob

Accept!

Reject!

Interactive Proofs [GMR85]

- Two new ingredients:
  - Randomness**:** verifier tosses coins, errs with some small probability
  - Interaction**:** rather than "reading" proof, verifier interacts with prover

- Classical proof systems lie in this framework: prover sends proof, verifier does not use randomness

Interactive Proofs [GMR85]

Interactive proof system for L is an interactive protocol (P, V)

- completeness: $x \in L$

  Pr[V accepts in (P, V)(x)] = 1

- soundness: $x \notin L, \forall P^*$

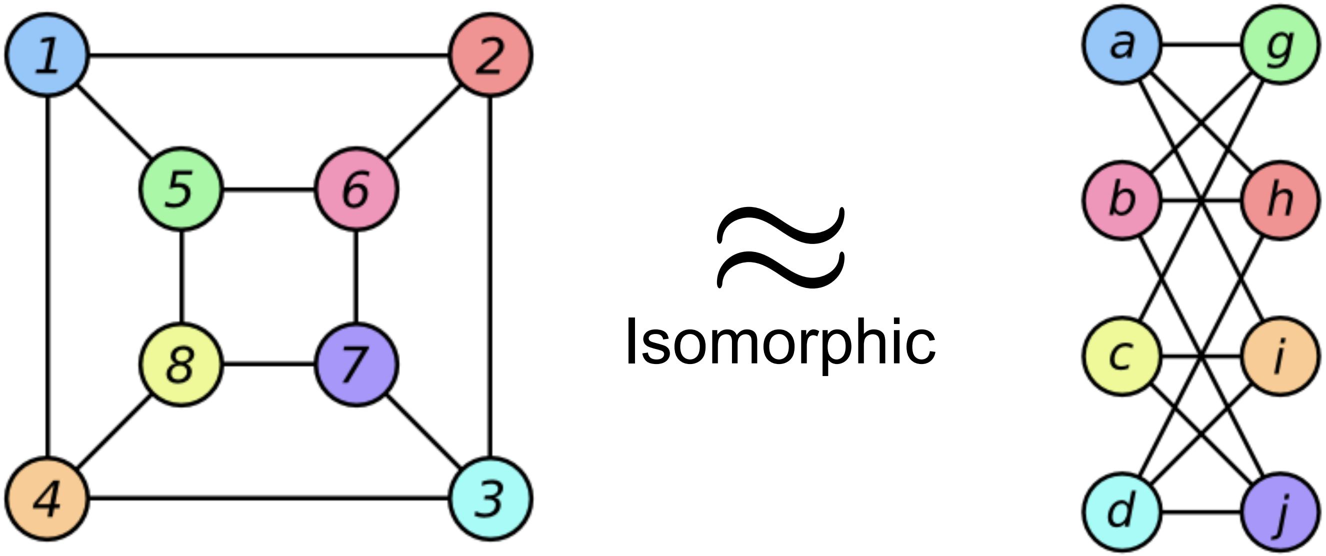
  Pr[V accepts in (P*, V)(x)] ≤ 1/2

- efficiency: V is p.p.t. machine

Repetition: can reduce error to any ε

Interactive Arguments: Soundness only against PPT machines

Interactive Proof for Graph Isomorphism



Graph $G_0 = (V_0, E_0)$
$V_0 = \{1, 2, \ldots, 8\}$
$E_0 = \{(1,2), (1.4), \ldots\}$

Graph $G_1 = (V_1, E_1)$
$V_1 = \{a, b, \ldots, j\}$
$E_1 = \{(a,g), (a,h), \ldots\}$

Isomorphic: Exists a mapping $\phi : V_0 \to V_1$ such that

$$(\alpha, \beta) \in E_0 \Leftrightarrow (\phi(\alpha), \phi(\beta)) \in E_1$$

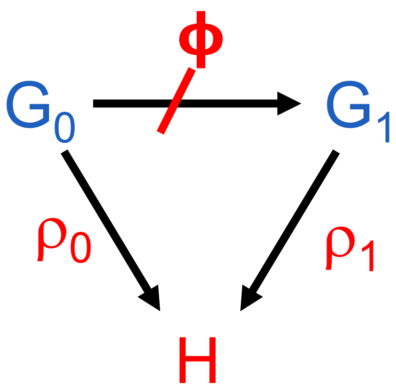
Interactive Proof for Graph Isomorphism

$$L = \{(G_0, G_1) \mid G_0 \approx G_1\}$$

$$G_0 \not\approx G_1$$

Prover

Alice



H →

b ∈ [0,1] ←

ρ_b →

Verifier

Bob

Accept if $\rho_b(G_b)=H$

Zero Knowledge Interactive Proofs

What is Knowledge?

Question as old as Humanity

Mostly studied in Philosophy: Epistemology

(also psychology, neuroscience, economics…)

Today, important in Computer Science

A Computational Approach to Knowledge [Goldwasser Micali 84]

2012 Turing Award Winners

"…for transformative work that laid the complexity-theoretic foundations for the science of cryptography, and in the process pioneered new methods for efficient verification of mathematical proofs in complexity theory"

A Computational Approach to Knowledge [Goldwasser Micali 84]

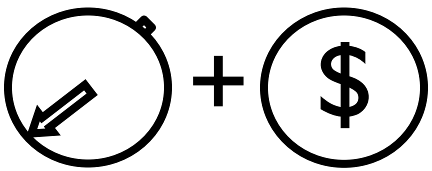
First in [GM84]: Probabilistic Encryption

Mature in [GMR85]: Zero-Knowledge + Proofs of knowledge

“I only know what I can feasibly compute”

Feasibly compute = PPT

Probabilistic Polynomial Time Turing Machines



Zero-Knowledge Proofs [GMR]

Prover
Alice

Verifier
Bob

Thank you Alice,
I believe X is true.
But I don't know why!

X= P vs NP

Completeness : P can convince V if X is true

Soundness: no (efficient) P* can convince V if X is not true

Zero Knowledge: no efficient V* learns anything more than validity of X

ZK Proof for Graph Isomo

Darn! I did not learn a thing

$G_0 \approx G_1$

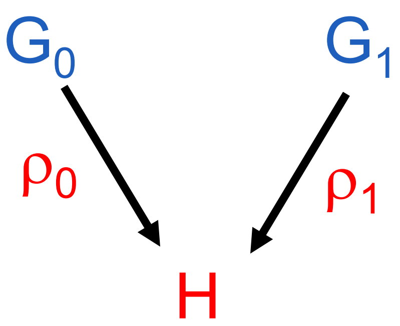
Prover
Alice

Verifier
Bob

H →

b ∈ [0,1] ←

ρ_b →

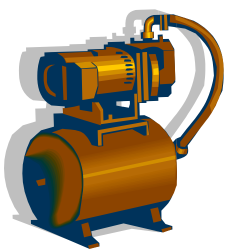


ZK Definition

$\forall$ PPT adversary verifier V^*, $\exists$ PPT *simulator* S such that

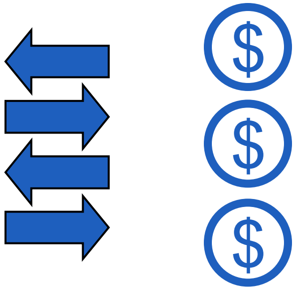
S-views $\approx$ V^*-views with Prover

Simulator

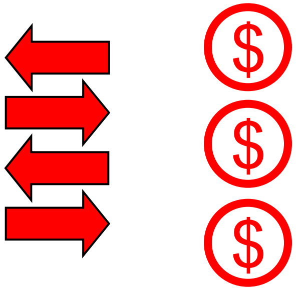


Prover

Verifier*



$\approx$



ZK Definition

$\forall$ PPT adversary verifier V^*, $\exists$ PPT *simulator* S such that

S-views $\approx$ *V**-views with Prover

ZK Rationale

V^* learns nothing that cannot be generated by V^* itself

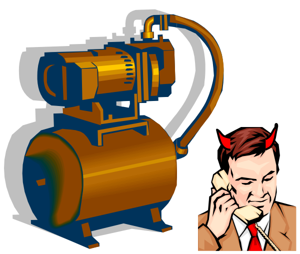
V^* itself = All Prob. Poly Time

ZK Definition

$\forall$ PPT adversary verifier V^*, $\exists$ PPT *simulator* S such that

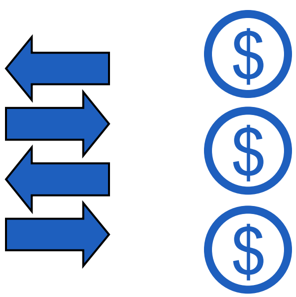
S-views $\approx$ V^*-views with Prover

Simulator

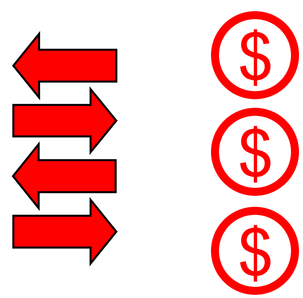


Prover

Verifier*



$\approx$



ZK as an instance* of MPC

NP language L with relation R

x,w

x

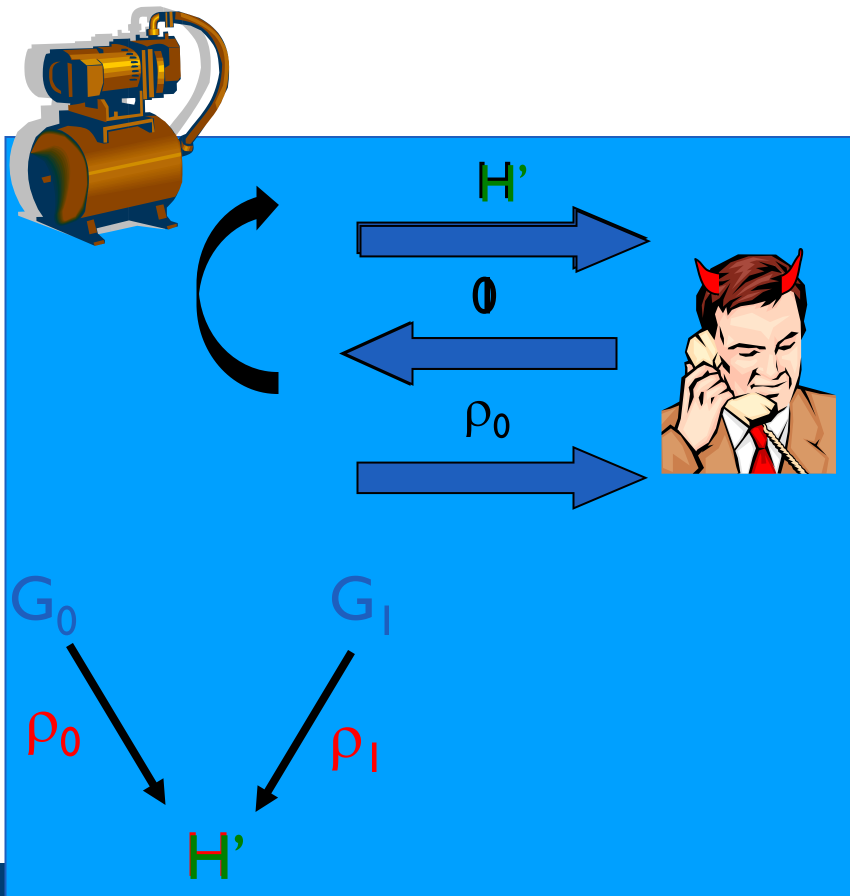
Securely Compute

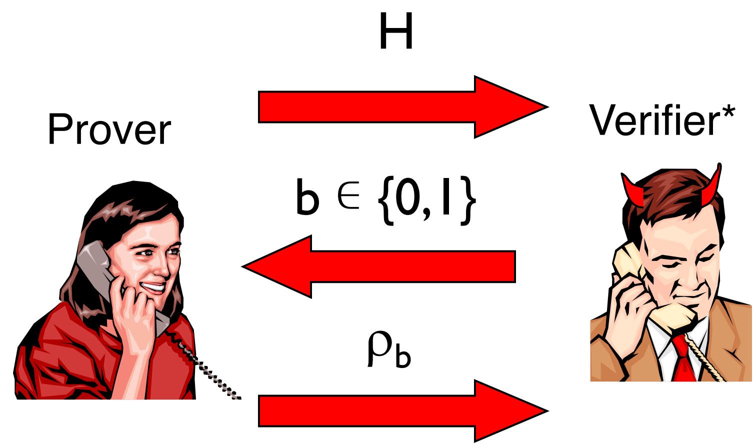
f(x,w) = R(x,w)

ZK Proof for Graph Isomorphism

$$G_0 \approx G_1$$

Simulator

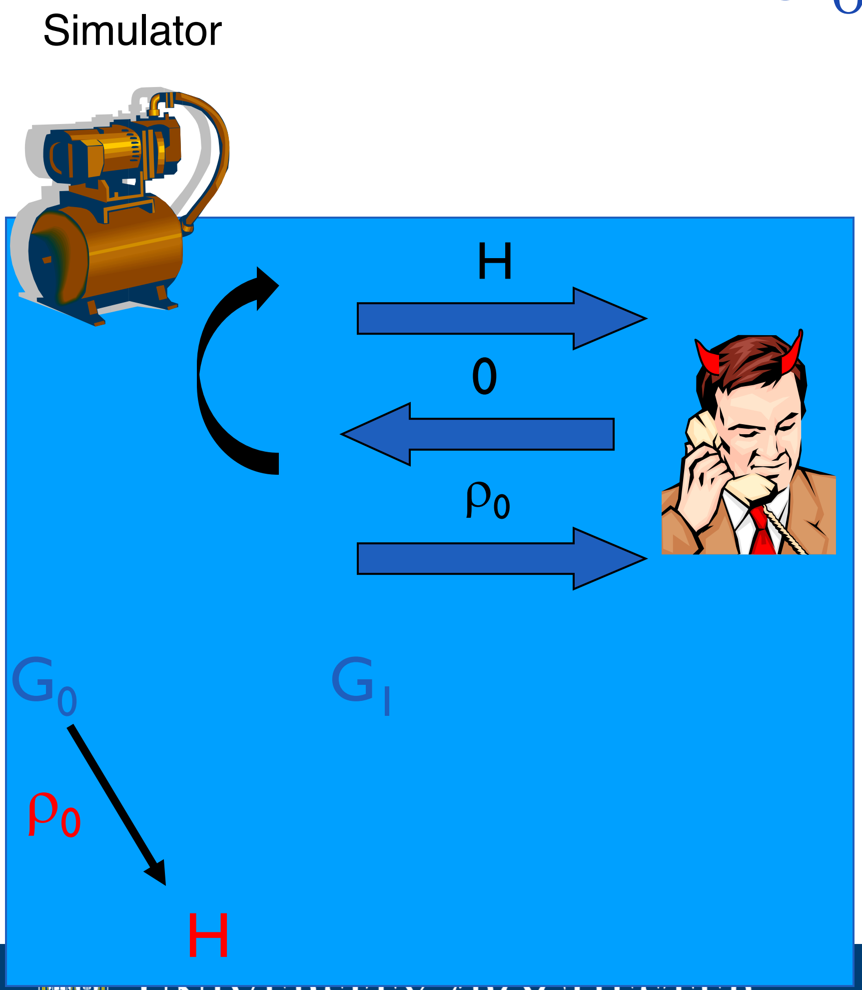




1. Choose G_0 or G_1 at random

ZK Proof for Graph Isomorphism

$$G_0 \approx G_1$$



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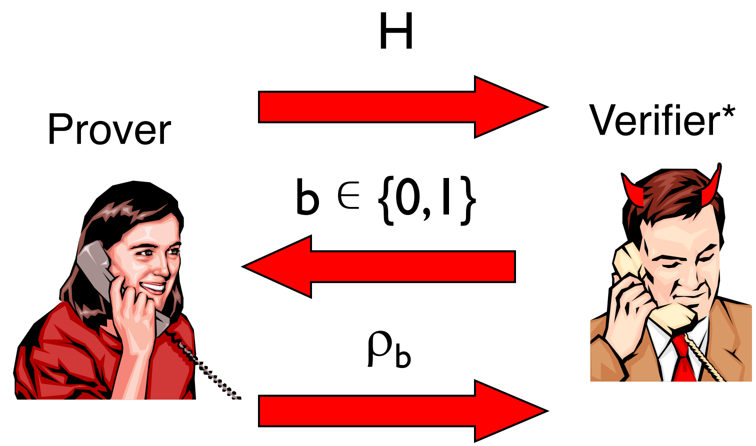


1. Choose G_0 or G_1 at random
2. Simulator will succeed w.p ½

What can you prove in ZK?

Can prove any classical proof in ZK [GMW86]
(a.k.a NP statements)

"Everything provable is provable in ZK" [BGGHKMR90]
(a.k.a languages in IP)

IP = PSPACE [S90,LFKN90]
PSPACE contains every language that is solvable with polynomial space

ZK for all of NP

Step 1: Construct a ZK Proof for an NP-complete language L_C

Step 2: Given any NP lang. L and instance x, compile* instance x to an instance x_C for L_C and use ZK Proof for $x_C \in L_C$

* compile via Karp reduction

Need Cryptographic Commitments

Commitment Scheme

The “digital analogue” of sealed envelopes.

v

Sender

Receiver

Com(v)

Commitment phase

Decommitment phase

d

v

Hiding: The commitment hides the committed value

Binding: The commitment can only open to one value

ZERO KNOWLEDGE FOR ~~ALL OF NP~~ Graph 3COL

Prover

Com(c(1)),…,Com(c(n)) →

← e=(i,j)

Open c(i) and c(j) →

Verifier

x = G(V,E)
w = c : V → {1,2,3}

x = G(V,E)
Accept iff c(i) ≠ c(j)

Completeness : Valid 3-Coloring satisfies c(i) ≠ c(j) for every edge e(i,j)

Soundness: Com() is **binding** ⇒ prover cannot change colors later

If G is **not 3 colorable**, prover caught on at least one edge. Occurs **w.p. 1/|E|**

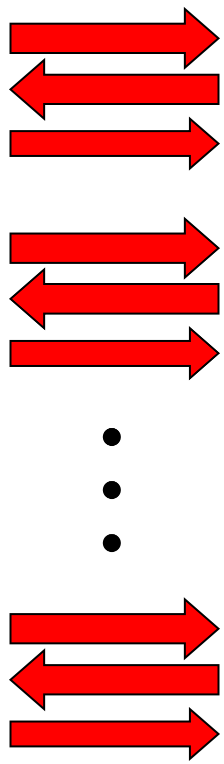
Zero Knowledge: Guess edge e(i,j) and give different colors for c(i) and c(j)

Constant s-soundness to negligible soundness

Repeat k log(1/s) times

Prover

Verifier



1 — Caught w.p. s

2 — Caught w.p. s

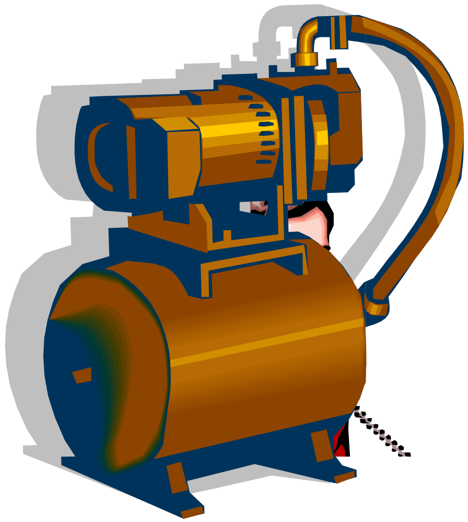
…

klog(1/s) — Caught w.p. s

Each rep. is indep. and soundness is $s^{k\log(1/s)} = 2^{-k}$

What about ZK property?

Repeat k log(1/s) times



Verifier

1 — Caught w.p. s

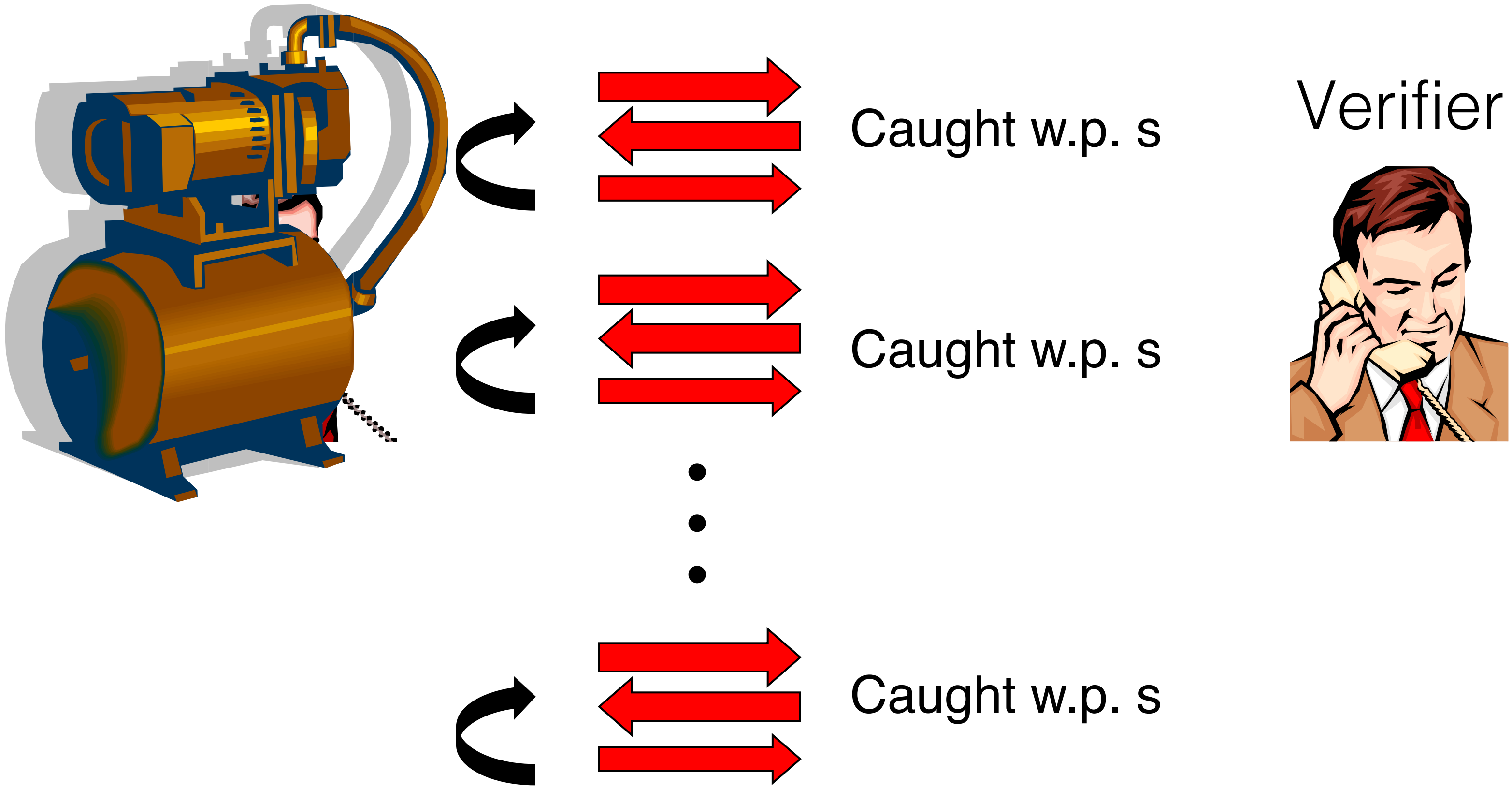
2 — Caught w.p. s

⋮

klog(1/s) — Caught w.p. s

Each rep. is indep. and soundness is $s^{k\log(1/s)} = 2^{-k}$

Can we repeat it in parallel?



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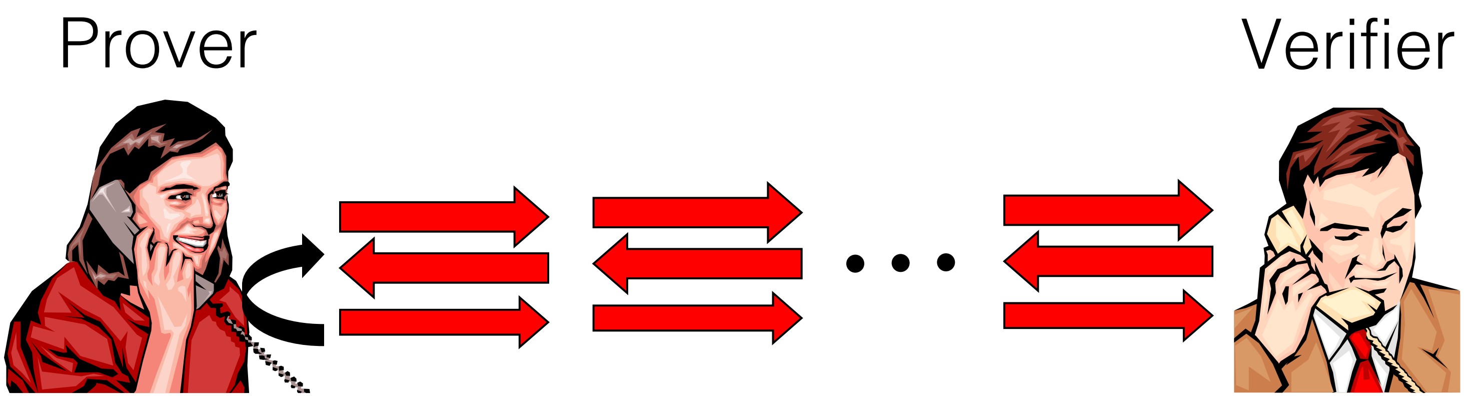


NO! Simulator's guess for all rep. are correct simultaneously only with probability 2^{-k}

Expected number of rewidings is 2^k

ZK for NP

ZK proof for Graph 3 Coloring [GMW86]

ZK proof for Hamiltonicity [Blum86]

ZK proof for SAT [BC87]

Theorem [BG+90]: Assume the existence of one-way functions. There exists a ZK proof for all of IP

ZK proof for any NP relation without using Karp reductions [IKOS07]

…more on Wednesday

Numerous Applications

- Boosting passive to active security
- Identification/ Authentication
- CCA secure encryption
- Resettable Security
- Bitcoins

Main Application: Active secure MPC

Compiling passive to active security when majority are dishonest

Passive adversaries (a.k.a. honest-but-curious) follow protocol instructions *to-the-word*

Passive-secure MPC protocol

↓

Coin Tossing | Zero Knowledge

Active-secure MPC protocol

Active adversaries (a.k.a malicious) arbitrarily deviate from protocol

Passive → Active: Enforce honest behavior

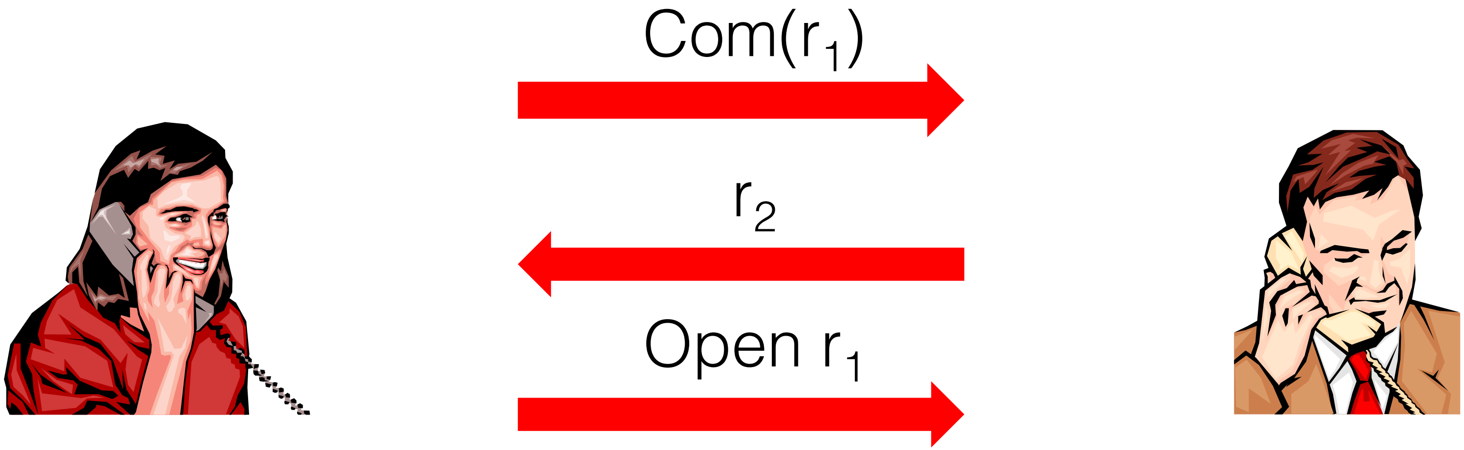
1. Force adversary to use a **fixed input** — Commitments

2. Force adversary to use a **uniform random tape** — Coin-tossing

3. Force adversary to **follow protocol instructions** exactly — Zero Knowledge

Coin Tossing

Goal: Fix random tape of every party

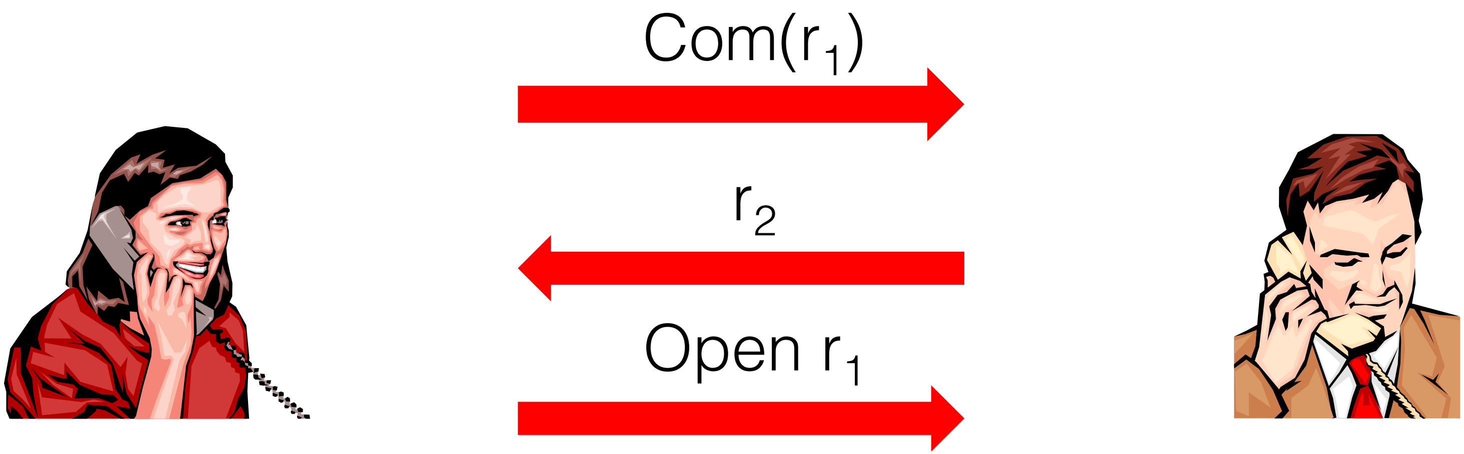


Output: $r_1 \oplus r_2$

Output: $r_1 \oplus r_2$

Augmented Coin Tossing: Fix Alice's tape

Goal: Alice's random tape is uniform.
Bob receives commitment to tape



Random tape = $r_1 \oplus r_2$

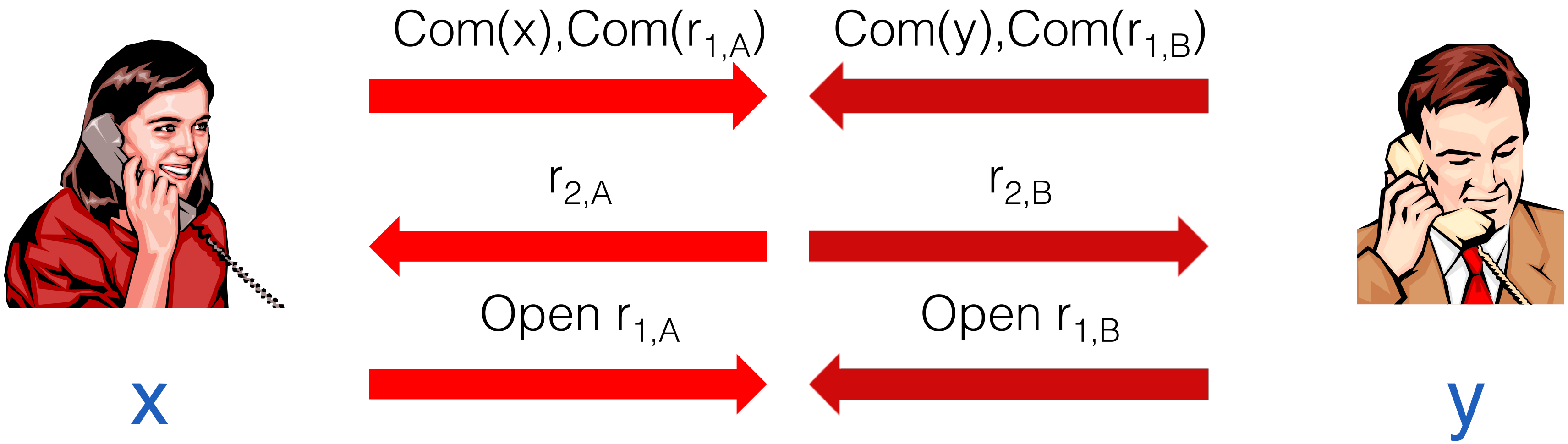
Output: $r_1 \oplus r_2$

Commitment to coin toss = $(Com(r_1), r_2)$

Output: $r_1 \oplus r_2$

Forcing good behavior

Preamble Phase:



After this stage, each party holds a commitment to the other party's input and random tape.

Main Insight: A protocol is a deterministic function of a party's input, random tape and series of incoming messages.

Forcing good behavior

Preamble Phase:

x

Com(x),Com($r_{1,A}$) →

← $r_{2,A}$

Open $r_{1,A}$ →

← Com(y),Com($r_{1,B}$)

$r_{2,B}$ →

← Open $r_{1,B}$

y

Execute passive protocol
Prove correctness of message every step

Forcing good b

Statement: Transcript

Witness: x, $r_{1,A}$ and rand. for $Com(x), Com(r_{1,A})$

Polytime Relation:

1. Check commitments correct w.r.t x, $r_{1,A}$
2. Check all messages generated according to honest Alice algorithm with input x and random tape $r_{1,A} \oplus r_{2,A}$

Preamble Phase:

$Com(x), Com(r_{1,A})$ → Cor ←

$r_{2,A}$ ←

Open $r_{1,A}$ → ←

⋮

$\mathbf{NxtMsg_i}$ →

x

"Correct": According to protocol specifications with input x and random tape $r_{1,A} \oplus r_{2,A}$

Expressible as an NP statement

Prove that $NxtMsg_i$ is "correct"

Caveat: Should not reveal witness!

Use ZK

Final Compilation (a.k.a GMW Paradigm)

Commit inputs and gen. rand tape

a_1

ZK Proof that a_1 is correct

b_1

ZK Proof that b_1 is correct

x

y

⋮

Execute passive secure protocol and give ZK Proof every step

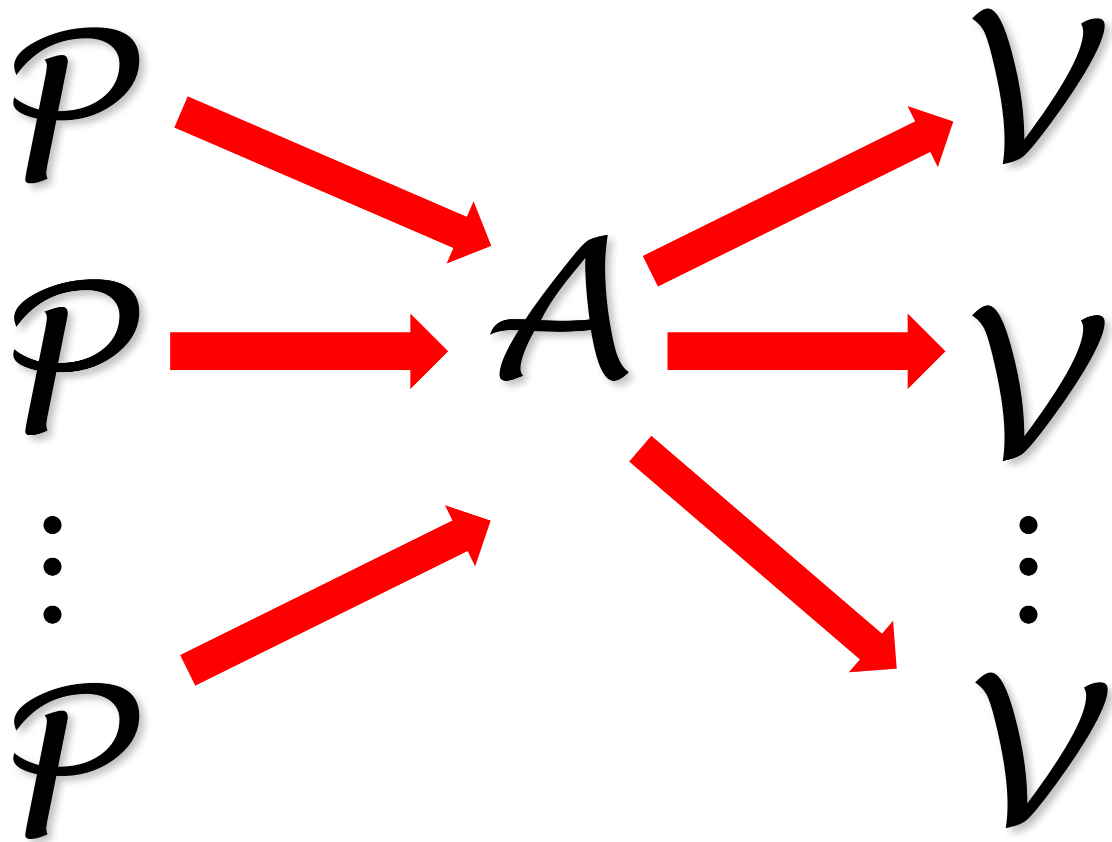
State-of-the-art for Active MPC

In theory, ZK Proofs allows compilation of passive to active security

In practice, use other techniques, eg, (cut-and-choose, MPC-in-the-head)

In fact, these other techniques have ZK implicit

Concurrency



Standard ZK is not secure in a concurrent setting

Zero Knowledge Proofs [GMR85]

Cornerstone of modern definitions of security

Techniques for arguing security

Fundamental cryptographic building block

Thank You