MPC Complexity

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The World of Functionalities

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  - Classical example: Communication Complexity [Yao]
  - MPC provides another lens to look at the complexity of functions

Complexity w.r.t. MPC

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  - Any other functionality can be reduced to OT
  - Under all notions of reduction (passive-secure, or UC secure)

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- The Cryptographic Complexity question:
  - Can F be reduced to G (for different reductions)?
  - G **complete** if everything reduces to G
  - F **trivial** if F reduces to everything (in particular, to NULL)

Quiz

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  - [x < y]

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  - (max(x,y), [x < y])

Complexity w.r.t. MPC

- Several notions of reductions
  - Passive, Active/Standalone or Active/UC
  - Information-theoretic (IT) or PPT
    - If PPT, also specify any computational assumptions used
- Will restrict to 2-party functionalities (mostly SFE)
  - In particular, omitting honest majority security

RECALL

Is MPC Possible?

- Can we securely realize <u>every</u> functionality?
- No & Yes!

Univ. Composable Angel-UC Standalone Passive	All subsets corruptible	Honest Majority
Computationally Unbounded (IT)	No	Yes
Computationally Bounded (PPT)	No Yes Yes Yes	

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> Yes means all are trivial.
> No is more interesting!

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every functionality?

In fact interesting: What computational hardness assumption makes it switch from No to Yes?

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Computationally Unbounded (IT)	No
Computationally Bounded (PPT)	No (Univ. Composable) Yes (Angel-UC) Yes (Standalone) Yes (Passive)

Trivial ones are *really* trivial
(called *Splittable*)

RECALL

An example

- Protocol:
  - Count down from 100
  - At each even round Alice announces whether her bid equals the current count; at each odd round Bob does the same
  - Stop if a party says yes
- Dutch flower auction

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Perfect Standalone Security
But doesn't compose!

Attack on
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    - Impossible to ensure this in IDEAL!

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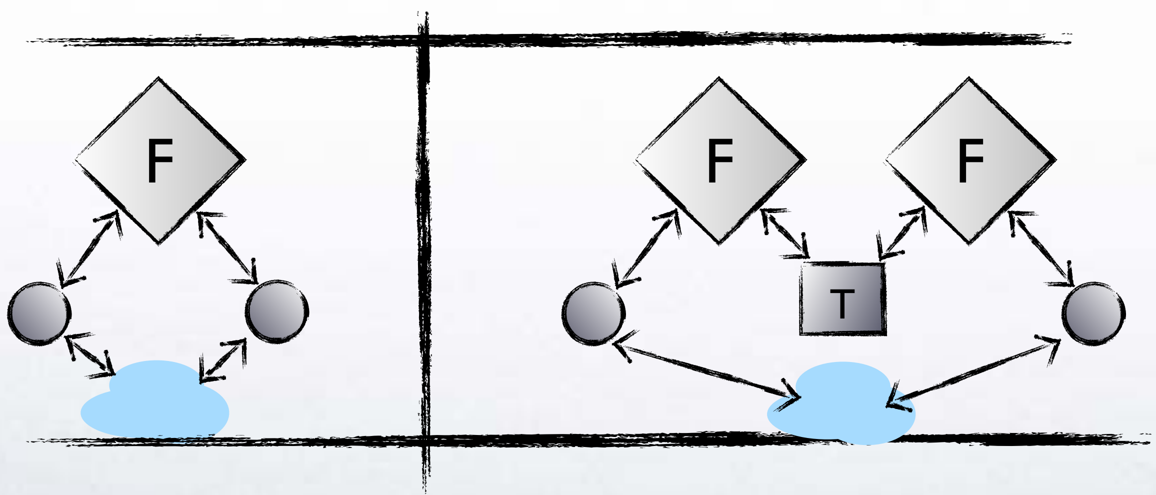
- Alice could get a result in one session, before running the other. But what should she submit as her input in the first one?
  - If a high bid, in trouble if she wins now, but Bob has a very low bid in the other session (which he must win).
  - If a low bid (so Bob may win with a low bid), in trouble if Bob has a high bid in the other session.

UC Triviality:
Splittability

- UC-trivial: "Splittable" [CKL'03,PR'08]
    - Literally trivial ones!



    - Extends to reactive, randomized functionalities, both PPT and IT

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Under sh-OT, everything else complete!

(Zero-One-Law)

IT Setting: Trivial Functionality

- Information-Theoretic Passive security
  - Deterministic SFE: Trivial ⇔ Decomposable

Decomposable Function

Decomposable

	1	3
0	1	3
2	2	3

	0	1
0	0	1
1	1	0

1	1	2
3	4	4

1	1	2	2
3	4	4	3

Undecomposable

	0	1
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1	0	1

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4	3	3

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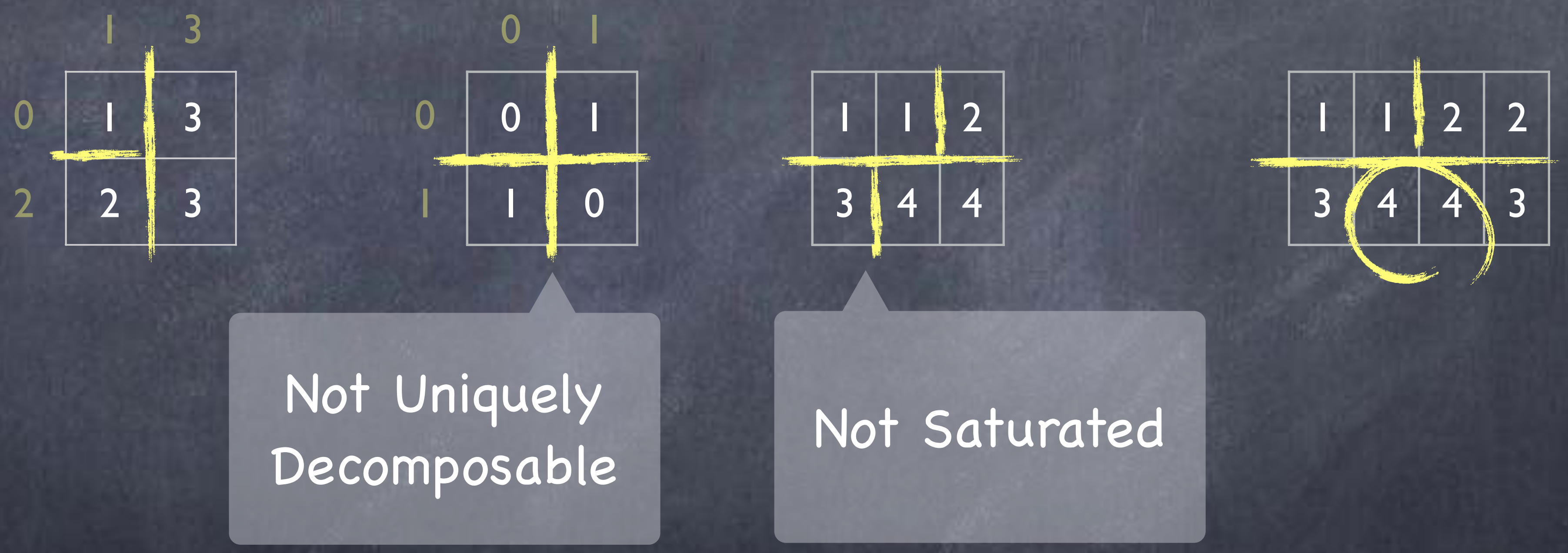

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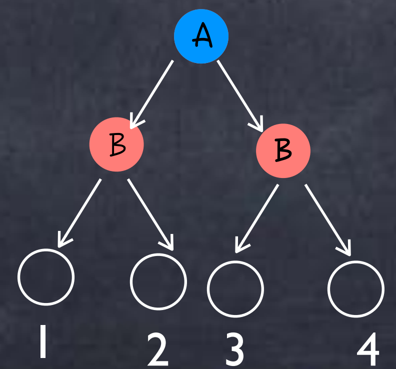

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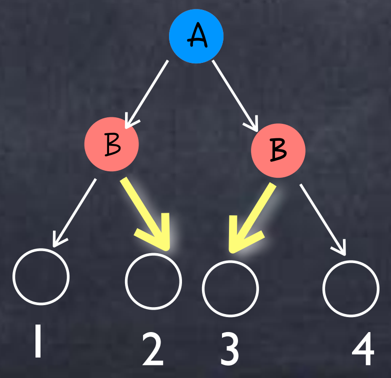
This strategy doesn't correspond to an input



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- Information-Theoretic UC security
  - Trivial ⇔ Splittable

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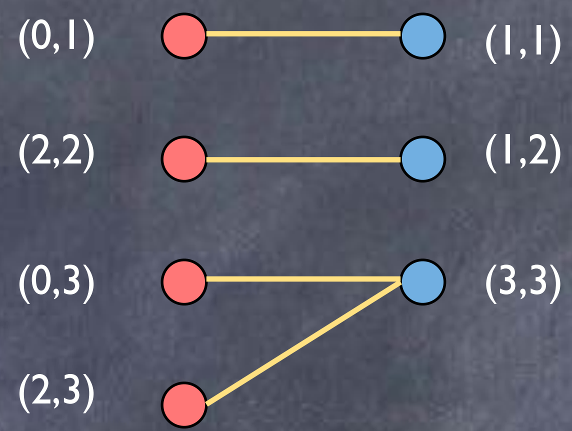
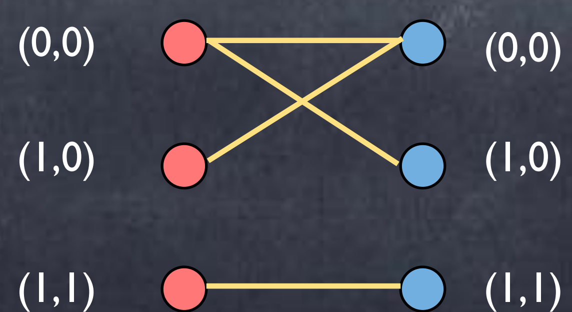


	0	1
0	0	0
1	0	1



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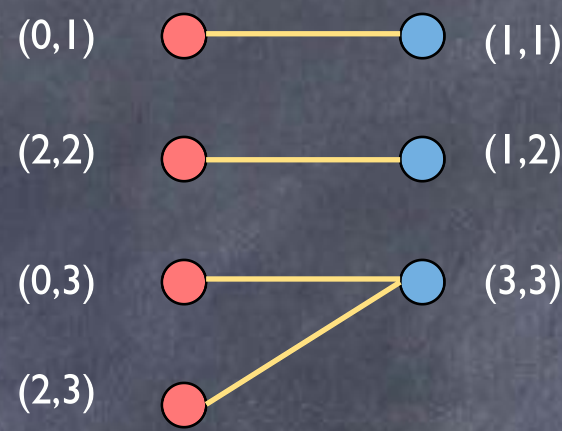
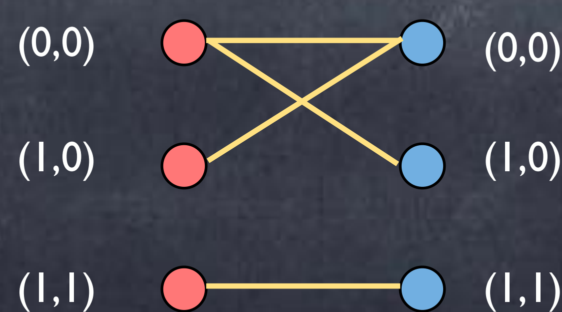


Simple:
Each connected component is a biclique

	0	1
0	0	0
1	0	1



IT Setting: Completeness

- Information-Theoretic Passive security
  - (Randomized) SFE: Complete ⇔ Not Simple
  - What is Simple?
    - Deterministic SFE: In the characteristic bipartite graph, each connected component is a biclique
    - More generally, using a weighted characteristic graph, with $w(u,v) = \Pr[\text{outputs} \mid \text{inputs}]$
      - Simple: $w(u,v) = w_A(u) \times w_B(v)$
    - "Isomorphic" to the "common information"

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  - What is the core of an SFE?
    - SFE obtained by removing "redundancies" in the input and output space

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1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
3	1	1	1	0

	0	1	2	3
0	0	1	2	3
1	1′	1	2	3
2	2′	2′	2	3
3	3′	3′	3′	3

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0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
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2	2	2	2	3
3	3	3	3	3

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0	0	0	0	0
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2	1	1	0	0
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  - (max(x,y), [x < y]) — Trivial (Passive and Standalone/Active)

	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
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	0	1	2	3
0	0	0	0	0
1	1	0	0	0
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1	1	0	0	0
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	0	1	2	3
0	0	1	2	3
1	1'	1	2	3
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Few Worlds Conjecture

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Few Worlds Conjecture

Many Worlds Conjecture

Summary

- 2-Party:
  - PPT, assuming sh-OT: 3 complexity classes. UC-trivial, UC-complete, All (= Passive/Standalone trivial/complete)
  - IT: Infinitely many complexity classes. Several open problems.
    - Computational assumptions related to collapse of classes in the PPT setting (so far OWF, shOT)
- m-Party (m>2):
  - Non-Honest-Majority: largely open

Quantitative Complexity

- Qualitative question: Does F reduce to G?
- Quantitative question: How many instances of G are needed to implement one instance of F (amortized)?
  - G-complexity of F
  - Upto constants, G-complexity remains the same for all complete G
  - "Cryptographic Complexity" of F
- Cryptographic Complexity is a lower bound on Circuit Complexity

Conclusion

- A detailed picture of deterministic 2-party SFE, under various MPC reductions
  - Completeness characterised for randomised SFE too
  - But complexity questions largely open for randomised SFE, m-party SFE for m > 2
- Computational hardness related to MPC reductions
  - We know that OWF is one of the "F reduces to G" assumptions, and sh-OT is the "maximal" assumption
  - Few Worlds Conjecture & Many Worlds Conjecture
- Quantitative Complexity
  - Crypto complexity is a lower bound on circuit complexity