Equivalence of SP, ONTO, ANON and median voting rule in single peaked domain

Theorem (Moulin 1980)

A strategyproof SCF f is onto and anonymous iff it is a median voter SCF.

Proof: $\Leftarrow$ median voter SCF is SP (previous theorem).

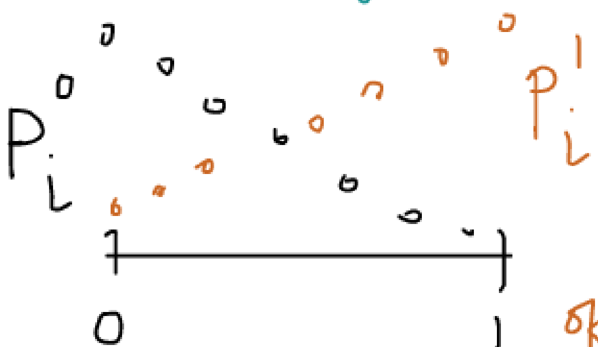
It is anonymous, if we permute the agents with peaks unchanged the outcome doesnot change.

It is onto, pick any arbitrary alternative a, put peaks of all players at a. The outcome will be a irrespective of the positions of the phantom peaks - since there are $(n-1)$ phantom peaks and n agent peaks.

$\Rightarrow$ Given, $f : \mathcal{S}^n \to A$ is SP, ANON, and ONTO.

define, P_i^0 : agent i's preference with peak at leftmost wrt $<$

P_i^1 : agent i's preference with peak at rightmost wrt $<$



The proof is constructive, we will construct the median voting rule (which needs the phantom peaks s.t. the outcome of an arbitrary f matches the outcome of the median SCF.

First, construct phantom peaks

$$y_j = f\big(\underbrace{P_1^0, P_2^0, \dots, P_{n-j}^0}_{(n-j)\text{ peaks leftmost}}, \underbrace{P_{n-j+1}^1, \dots, P_n^1}_{j\text{ peaks rightmost}}\big), \quad j = 1, \dots, n-1$$

Which agents have which peaks does not matter because of anonymity.

Claim: $y_j \leq y_{j+1}$, $j = 1, \dots, n-2$, i.e., peaks are non-decreasing.

Proof: $y_{j+1} = f(P_1^0, P_2^0, \dots, P_{n-j}^1, P_{n-j+1}^1, \dots, P_n^1)$

Due to SP, $y_j \, P_{n-j}^0 \, y_{j+1}$ — or they are same, but P_{n-j}^0 is single peaked with peak at 0, hence $y_j \leq y_{j+1}$. □

Consider an arbitrary profile, $P = (P_1, P_2, \dots, P_n)$, $P_i(1) = p_i$ (the peaks).

Claim: Suppose f satisfies SP, ONTO, ANON, then

$$f(P) = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1}).$$

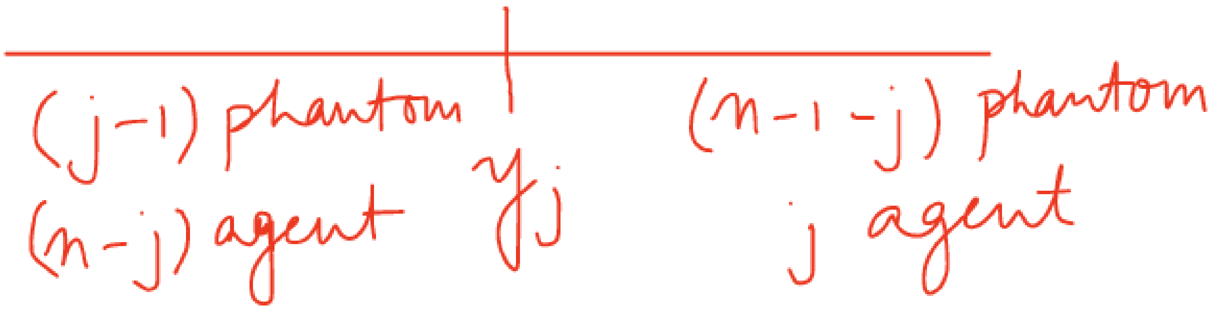
WLOG, can assume $p_1 \leq p_2 \leq \dots \leq p_n$ due to ANON.

also say, $a = \text{median}(p_1, \dots, p_n, y_1, \dots, y_{n-1})$

Case 1: a is a phantom peak

Say $a = y_j$, for some $j \in \{1, 2, \dots, n-1\}$.

This is a median of $2n-1$ points, of which $(j-1)$ phantom peaks lie on the left (see the claim before). Rest $(n-j)$ points are agent peaks.



Hence, $p_1 \leq \dots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \dots \leq p_n$.

Use a similar transformation as we used earlier

$$f(P_1^0, P_2^0, \dots, P_{n-j}^0, P_{n-j+1}^1, \dots, P_n^1) = y_j \quad \text{(definition)}$$

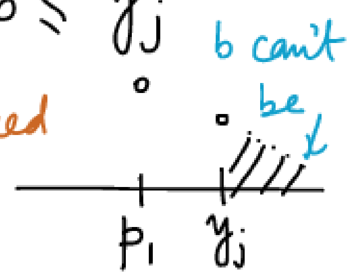
$$f(P_1, P_2^0, \dots, P_{n-j}^0, P_{n-j+1}^1, \dots, P_n^1) = b \quad \text{(say)}$$

By SP, $y_j \, P_1^0 \, b \Rightarrow y_j \leq b$

again by SP, $b \, P_1 \, y_j$, but $p_1 \leq y_j \overset{\uparrow}{\Rightarrow} b \leq y_j$

single peaked

hence $b = y_j$



repeat this argument for first $(n-j)$ agents to get

$$f(P_1, P_2, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P'_n) = y_j$$

now consider

$$f(P_1, P_2, \ldots, P_{n-j}, P'_{n-j+1}, \ldots, P_n) = b \text{ (say)}$$

apply very similar argument

$$\left.\begin{array}{l} y_j \; P'_n \; b \;\Rightarrow\; b \leq y_j \\ \\ b \; P_n \; y_j \text{ and } y_j \leq P_n \;\Rightarrow\; y_j \leq b \end{array}\right\} \; b = y_j$$

Hence $f(P_1 \ldots, P_n) = y_j$.