Claim: Suppose f satisfies SP, ONTO, ANON, then

$$f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}).$$

Case 1: a is a phantom peak - proved

Case 2: a is an agent peak

We will prove this for 2 players. The general case repeats this argument.

Claim: $N = \{1, 2\}$, let P and P' be such that

$P_i(1) = P_i'(1)$, $\forall i \in N$. Then $f(P) = f(P')$.

Proof: Let $a = P_1(1) = P_1'(1)$, and $P_2(1) = P_2'(1) = b$.

$f(P) = x$ and $f(P_1', P_2) = y$

Since f is SP, $x P_1 y$ and $y P_1' x$

Since peaks of P_1 and P_1' are the same, if x, y are on the same side of the peak, they must be the same, as the domain is single peaked.

The only other possibility is that x and y fall on different sides of the peak. We show that this is impossible.

WLOG $x < a < y$ and $a < b$

f is SP+ONTO $\Leftrightarrow$ f is SP+PE

PE requires $f(P) \in [a, b]$, but $f(P) = x < a$ $\rightarrow\leftarrow$

now repeat this argument for $(P_1', P_2) \to (P_1', P_2')$ $\square$

Profile: $(P_1, P_2) = P$, $P_1(1) = a$, $P_2(1) = b$

y_1 is the phantom peak.

by assumption, median (a, b, y_1) is an agent peak

WLOG assume the median is a.

Assume for contradiction $f(P) = c \neq a$.

By PE, c must be within a and b. We have two cases to consider: $b < a < y_1$ and $y_1 < a < b$.

Case 2.1: $b < a < y_1$, by PE $c < a$

construct P_1' s.t. $P_1'(1) = a = P_1(1)$

and $y_1 P_1' c$ (possible since they are on different sides of a)

by the earlier claim, $f(P) = c \Rightarrow f(P_1', P_2) = c$.

now consider the profile (P_1', P_2)

peak at the rightmost

$P_2(1) = b < y_1 \leq P_1'(1)$, hence the median of $\{b, y_1, P_1'(1)\}$ is y_1 (which is a phantom peak, hence case 1 applies).

$f(P_1', P_2) = y_1$.

But $y_1 \, P_1' \, c$ (by construction) and $f(P_1', P_2) = c$

agent 1 manipulates $P_1' \rightarrow P_1'$, contradiction to f being SP.

Case 2·2: $y_1 < a < b$, PE $\Rightarrow a < c$

construct P_1' s.t. $P_1'(1) = a = P_1(1)$ and $y_1 \, P_1' \, c$

$f(P_1', P_2) = c$ (by claim)

Consider (P_1^0, P_2), $P_1^0(1) \leq y_1 < b \Rightarrow f(P_1^0, P_2) = y_1$

but $y_1 \, P_1' \, c$, hence manipulable by agent 1.

This completes the proof for two agents (case 2). For the generalization to n players, see Moulin (1980) "On strategyproofness and single peakedness".