CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3

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Contents

Matrix games: *two player zero-sum* games

A special class with certain nice **security** and **stability** properties

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Definition (Two player zero-sum games)

A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$

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Question

Why called **matrix** game?

Matrix games: *two player zero-sum* games

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Question

Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

	Player 2	
Player 1	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

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$$\Longrightarrow$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

		L	R	maxmin
Player 1	L	-1	1	-1
	R	1	-1	-1
	minmax	1	1	

Example: Penalty shoot game

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		L	R
Player 1	L	−1, 1	1, −1
	R	1, −1	−1, 1

$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

Player 2's ***maxmin*** *value is the* ***minmax*** *value of this matrix*

		L	R	maxmin
Player 1	L	−1	1	−1
	R	1	−1	−1
	minmax	1	1	

Another example

		Player 2		
		L	C	R
Player 1	T	$3, -3$	$-5, 5$	$-2, 2$
	M	$1, -1$	$4, -4$	$1, -1$
	B	$6, -6$	$-3, 3$	$-5, 5$

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		L	C	R	maxmin
Player 1	T	3	-5	-2	-5
	M	1	4	1	1
	B	6	-3	-5	-5
	minmax	6	4	1	

Two examples together

Player 1

	L	R	maxmin
L	−1	1	−1
R	1	−1	−1
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Player 1

	L	C	R	maxmin
T	3	−5	−2	−5
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Question

What are the PSNEs for the above games?

Two examples together

		L	R	maxmin
Player 1	L	−1	1	−1
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Question

What are the PSNEs for the above games?

Answer

Left: no PSNE; **Right:** (M,R)

Saddle point

Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

Saddle point

Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

Question

What are the saddle points for the previous two games?

Saddle point

Player 1

	L	R
L	-1	1
R	1	-1

Player 1

	L	C	R
T	3	-5	-2
M	1	4	1
B	6	-3	-5

Saddle point

		L	R
Player 1	L	-1	1
	R	1	-1

		L	C	R
Player 1	T	3	-5	-2
	M	1	4	1
	B	6	-3	-5

Answer

For the first example: no saddle point, for the second: (M,R)

Saddle point

$$\text{Player 1}\quad \begin{array}{c|c|c|}
 & L & R \\ \hline
L & -1 & 1 \\ \hline
R & 1 & -1 \\ \hline
\end{array}
\qquad\qquad
\text{Player 1}\quad \begin{array}{c|c|c|c|}
 & L & C & R \\ \hline
T & 3 & -5 & -2 \\ \hline
M & 1 & 4 & 1 \\ \hline
B & 6 & -3 & -5 \\ \hline
\end{array}$$

Answer

For the first example: no saddle point, for the second: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.

Saddle point and PSNE

Proof.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE. □

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Consider **maxmin** and **minmax** values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \qquad \textbf{maxmin}$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \qquad \textbf{minmax}$$

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Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE. ☐

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Question

How are the maxmin and minmax values related?

Relationship of the security values

Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

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$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2),\ \forall s_1, s_2, \qquad \text{definition of min}$$

□

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For matrix games $\overline{v} \geqslant \underline{v}$.

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$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2, \qquad \text{definition of min}$$

$$\Rightarrow \max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \ \forall s_2 \in S_2 \qquad \text{RHS was dominated for each } s_1$$

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Relationship of the security values

Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

Proof.

$$\begin{aligned} u(s_1,s_2) &\geqslant \min_{t_2\in S_2} u(s_1,t_2),\ \forall s_1,s_2, && \text{definition of min}\\ \Rightarrow \max_{t_1\in S_1} u(t_1,s_2) &\geqslant \max_{t_1\in S_1}\min_{t_2\in S_2} u(t_1,t_2),\ \forall s_2 \in S_2 && \text{RHS was dominated for each } s_1\\ \Rightarrow \min_{t_2\in S_2}\max_{t_1\in S_1} u(t_1,t_2) &\geqslant \max_{t_1\in S_1}\min_{t_2\in S_2} u(t_1,t_2) && \text{RHS was a constant} \end{aligned}$$

□

Contents

Earlier examples and security values

Player 1

	L	R	maxmin
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minmax	1	1	

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PSNE does not exist

Earlier examples and security values (contd.)

		L	C	R	maxmin
Player 1	T	3	−5	−2	−5
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Earlier examples and security values (contd.)

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PSNE exists

PSNE Theorem

Define the following strategies

$$s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2),$$ **maxmin strategy of player 1**

$$s_2^* \in \arg \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2),$$ **minmax strategy of player 2**

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$$s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2),$$

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$$s_2^* \in \arg \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2),$$

minmax strategy of player 2

Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$*, where* s_1^* *and* s_2^* *are maxmin and minmax strategies for players* 1 *and* 2 *respectively.*

Corollary: (s_1^*, s_2^*) is a PSNE

Proof of the PSNE Theorem

Proof

($\implies$ **)** let (s_1^*, s_2^*) is a PSNE $\implies \overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax

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Since (s_1^*, s_2^*) is a PSNE, $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$.

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$$\begin{aligned} \implies u(s_1^*, s_2^*) &\geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\ &\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy} \\ &= \overline{v} \end{aligned}$$

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$$\begin{aligned}& u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*) \\ \implies & u(s_1^*, s_2^*) = \overline{v} = \underline{v}\end{aligned}$$

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Also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* respectively.

Proof of the PSNE Theorem (contd.)

Proof (contd.)

($\Longleftarrow$) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax $\implies$ (s_1^*, s_2^*) is a PSNE

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Similarly, we can show $u(s_1, s_2^*) \leqslant v, \ \forall s_1 \in S_1$
But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

Contents

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player

		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player

		Player 2	
		L	R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
	$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player

		Player 2	
		$\frac{4}{5}$ L	$\frac{1}{5}$ R
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- Consider a finite set A, define

$$\Delta A = \{p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1\}$$

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- Consider a finite set A, define

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- Mixed strategy set of player 1: $\Delta S_1 = \Delta\{L, R\}$, $(\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$

Mixed Strategies (contd.)

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- We are discussing non-cooperative games, the players choose their strategies independently
- The joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$
- Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \; u_i(s_1, s_2, \ldots, s_n)$$

Mixed Strategies (contd.)

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- $\sigma_i \in \Delta S_i$, i.e. , $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- We are discussing non-cooperative games, the players choose their strategies independently
- The joint probability of player 1 picking s_1 and player 2 picking s_2 = $\sigma_1(s_1)\sigma_2(s_2)$
- Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \; u_i(s_1, s_2, \ldots, s_n)$$

- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies

Mixed Strategies (contd.)

- **Notation**: σ_i is a mixed strategy of player i
- $\sigma_i \in \Delta S_i$, i.e. , $\sigma_i : S_i \to [0, 1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- We are discussing non-cooperative games, the players choose their strategies independently
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- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.

Example

		Player 2	
		$\frac{4}{5}$ L	$\frac{1}{5}$ R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
	$\frac{1}{3}$ R	$1, -1$	$-1, 1$

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Player 2

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$u_1(\sigma_1, \sigma_2)$

Example

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$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1)$$

Example

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Player 1	$\frac{4}{5}$ L	$\frac{1}{5}$ R
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Mixed Strategies Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A **mixed strategy Nash equilibrium (MSNE)** is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$$

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Question

Relation between **PSNE** and **MSNE**?

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Answer

PSNE $\implies$ MSNE

An Alternative Definition

Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$*, is an* ***MSNE*** *if and only if* $\forall s_i \in S_i$ *and* $\forall i \in N$

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Proof.

($\Rightarrow$): The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE

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Examples of MSNE

Question

Is the mixed strategy profile an **MSNE**?

		Player 2	
		$\frac{4}{5}$ L	$\frac{1}{5}$ R
Player 1	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
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- To answer this, we need to show that there does not exist any better mixed strategy for the player

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- Expected utility of player 2 from $L = 2/3 \cdot 1 + 1/3 \cdot (-1) = 1/3$
- Expected utility of player 2 from $R = 2/3 \cdot (-1) + 1/3 \cdot 1 = -1/3$

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- $\Rightarrow$ the current profile is **not** an MSNE

Examples of MSNE

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Is the mixed strategy profile an **MSNE**?

Player 2

		$\frac{4}{5}$ L	$\frac{1}{5}$ R
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- Expected utility will increase if some probability is transferred from R to L
- $\Rightarrow$ the current profile is **not** an MSNE
- Some balance in the utilities is needed

Examples of MSNE

Question

Is the mixed strategy profile an **MSNE**?

Player 2

		$\frac{1}{2}$ L	$\frac{1}{2}$ R
Player 1	$\frac{1}{2}$ L	$-1, 1$	$1, -1$
	$\frac{1}{2}$ R	$1, -1$	$-1, 1$

- Expected utility will increase if some probability is transferred from R to L
- $\Rightarrow$ the current profile is **not** an MSNE
- Some balance in the utilities is needed
- **Does there exist any improving mixed strategy?**

Contents

How to find an MSNE

Definition (Support of mixed strategy/probability distribution)

For mixed strategy σ_i, the subset of strategy set of i on which σ_i has a positive mass is called the **support** of σ_i and is denoted by $\delta(\sigma_i)$. Formally, $\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.

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Using the definition of support, here is a characterization of MSNE

Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is an MSNE iff*[a] $\forall i \in N$

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[a] This is a shorthand for 'if and only if'.

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A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ *is an MSNE iff*[a] $\forall i \in N$

1. $u_i(s_i, \sigma_{-i}^*)$ *is identical* $\forall s_i \in \delta(\sigma_i^*)$,

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Implication

Consider Penalty Shoot Game

		Goalkeeper	
		L	R
Shooter	L	$-1, 1$	$1, -1$
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Case 1: support profile $(\{L\}, \{L\})$: for player 1 , $s_1' = R$ – violates condition 2

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Case 1: support profile $(\{L\}, \{L\})$: for player 1 , $s'_1 = R$ – violates condition 2

Case 2: support profile $(\{L, R\}, \{L\})$ – symmetric for the other case

For Player 1, the expected utility has to be the same for L and R - not possible – violates condition 1

Implication

Case 3: support profile $(\{L, R\}, \{L, R\})$: condition 2 is vacuously satisfied

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For condition 1, let player 1 chooses L w.p. p and player 2 choose L w.p. q

For player 1:

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q)) \ \Rightarrow (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \ \Rightarrow q = \frac{1}{2}$$

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For player 2:

$$u_2((p, 1-p), L) = u_2((p, 1-p), R) \ \Rightarrow p = \frac{1}{2}$$

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For player 2:

$$u_2((p,1-p),L) = u_2((p,1-p),R) \;\Rightarrow p = \frac{1}{2}$$

MSNE =

$$\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

Exercises

Player 2

		F	C
Player 1	F	2, 1	0, 0
	C	0, 0	1, 2

Player 2

		F	C	D
Player 1	F	2, 1	0, 0	1, 1
	C	0, 0	1, 2	2, 0

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A mixed strategy profile is an MSNE iff $\forall i \in N$

1. $u_i(s_i, \sigma^*_{-i})$ *is identical* $\forall s_i \in \delta(\sigma^*_i)$,
2. $u_i(s_i, \sigma^*_{-i}) \geqslant u_i(s'_i, \sigma^*_{-i}), \forall s_i \subseteq \delta(\sigma^*_i), s'_i \notin \delta(\sigma^*_i)$.

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Observations:

- $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$
  maximizing w.r.t. a distribution ⇔ whole probability mass at max

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  maximizing w.r.t. a distribution ⇔ whole probability mass at max
- If $(\sigma^*_i, \sigma^*_{-i})$ is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma^*_{-i}) = \max_{s_i \in \delta(\sigma^*_i)} u_i(s_i, \sigma^*_{-i})$$

the maximizer must lie in $\delta(\sigma^*_i)$ – if not, then put all probability mass on that $s'_i \notin \delta(\sigma^*_i)$ that has the maximum value of the utility – $(\sigma^*_i, \sigma^*_{-i})$ is not a MSNE

Proof of MSNE Characterization Theorem

$(\Rightarrow)$ Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \tag{1}$$

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By definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \tag{2}$$

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Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: **proves condition 1**

Proof (contd.)

For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t. $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

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$(\Leftarrow)$ Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{aligned} &\text{Let} \quad u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*) && \textbf{condition 1} \\ &\text{Note} \quad m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) && \textbf{condition 2} \end{aligned}$$

Proof (contd.)

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), \qquad \textbf{by definition of } \delta(\sigma_i^*)$$

Proof (contd.)

$$\begin{aligned} u_i(\sigma_i^*, \sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), && \textbf{by definition of } \delta(\sigma_i^*) \\ &= m_i(\sigma_{-i}^*) && \textbf{previous conclusion} \end{aligned}$$

Proof (contd.)

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This proves the sufficient direction. The result yields an algorithmic way to find MSNE

Contents

MSNE characterization theorem to algorithm

Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

MSNE characterization theorem to algorithm

Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

The total number of supports of $S_1 \times S_2 \times S_3 \cdots \times S_n$ is
$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

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For every support profile $X_1 \times X_2 \times \cdots X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

Program

$$w_i = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

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Remarks on the algorithm

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- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs] [1]

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MSNE and Dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Is there a dominated strategy in this game? Domination can be via mixed strategies too

		Player 2	
		L	R
Player 1	T	4, 1	2, 5
	M	1, 3	6, 2
	B	2, 2	3, 3

MSNE and Dominance

Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

Contents

- Matrix games
- Relation between **maxmin** and PSNE
- Mixed Strategies
- Mixed Strategy Nash Equilibrium
- Find MSNE
- MSNE Characterization Theorem Proof
- Algorithm to find MSNE
- Existence of MSNE

Existence of MSNE

Definition (Finite Games)

A game is said to be **finite** when the number of players is finite, and each player has a finite set of strategies.

Existence of MSNE

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Proof requires a few tools and a result from real analysis. Proof is separately given in the course webpage.

Existence of MSNE

Some background for understanding the proof.

- A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in S$.

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A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T : S \to S$, is **continuous** then T has a fixed point, i.e., $\exists\, x^* \in S$ s.t. $T(x^*) = x^*$.

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