

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3

Swaprava Nath

Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम् Knowledge is the supreme goal

Contents

► Matrix games

- ▶ Relation between **maxmin** and PSNE
- ► Mixed Strategies
- Mixed Strategy Nash Equilibrium
- ► Find MSNE
- ▶ MSNE Characterization Theorem Proof
- ► Algorithm to find MSNE
- ► Existence of MSNE





Definition (Two player zero-sum games)

A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$

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Question

Why called **matrix** game?

Definition (Two player zero-sum games)

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Why called **matrix** game?

Answer

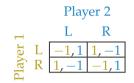
Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix



Question

Example: Penalty shoot game

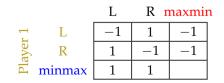




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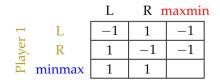


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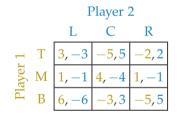


Player 2's maxmin value is the minmax value of this matrix



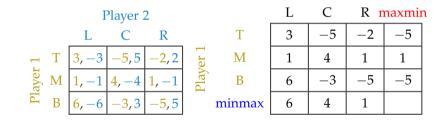
Another example





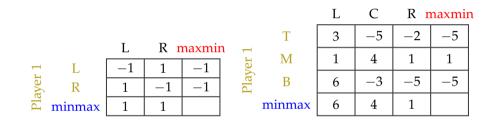
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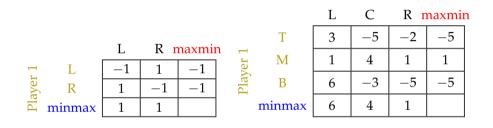
Two examples together





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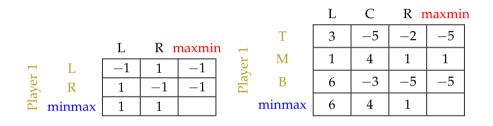


Question

What are the PSNEs for the above games?

Two examples together





Question

What are the PSNEs for the above games?

Answer

Left: no PSNE; **Right:** (M,R)





Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2





Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

Question

What are the saddle points for the previous two games?

Saddle point





Saddle point





Answer

For the first example: no saddle point, for the second: (M,R)

Saddle point





Answer

For the first example: no saddle point, for the second: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.



Proof.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \ge u(s_1, s_2^*)$, $\forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \le u(s_1^*, s_2)$, $\forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \ge u_1(s_1, s_2^*)$, $\forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \ge u_2(s_1^*, s_2)$, $\forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE.



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Consider maxmin and minmax values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$



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minmax

Question

How are the maxmin and minmax values related?

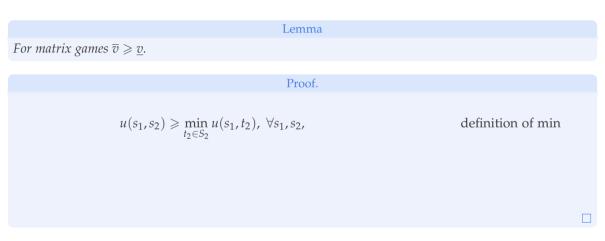
Relationship of the security values

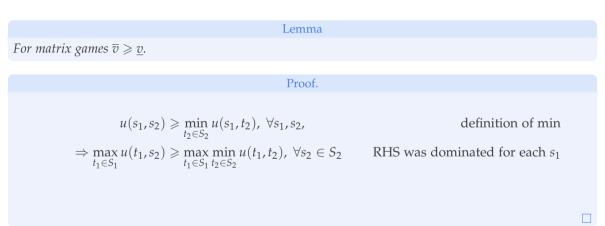


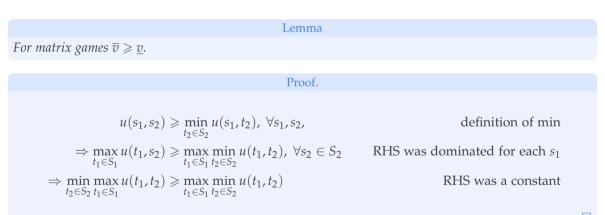
Lemma

For matrix games $\overline{v} \ge \underline{v}$.

Relationship of the security values







Contents

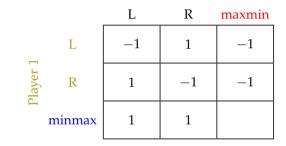


Matrix games

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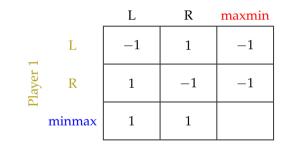
Earlier examples and security values





Earlier examples and security values

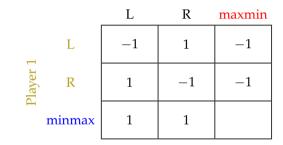




$$\overline{v} = 1 > -1 = \underline{v}$$

Earlier examples and security values





 $\overline{v} = 1 > -1 = \underline{v}$ PSNE does not exist

Earlier examples and security values (contd.)



| | | L | С | R | maxmin |
|----------|--------|---|----|----|--------|
| Player 1 | Т | 3 | -5 | -2 | -5 |
| | Μ | 1 | 4 | 1 | 1 |
| | В | 6 | -3 | -5 | -5 |
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Earlier examples and security values (contd.)



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 $\overline{v} = 1 = \underline{v}$ PSNE exists



Define the following strategies

$$\begin{split} s_1^* &\in \arg \, \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \\ s_2^* &\in \arg \, \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2), \end{split}$$

maxmin strategy of player 1

minmax strategy of player 2



Define the following strategies

 $s_{1}^{*} \in \arg \max_{s_{1} \in S_{1}} \min_{s_{2} \in S_{2}} u(s_{1}, s_{2}),$ $s_{2}^{*} \in \arg \min_{s_{2} \in S_{2}} \max_{s_{1} \in S_{1}} u(s_{1}, s_{2}),$ maxmin strategy of player 1

minmax strategy of player 2

Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively.

Corollary: (s_1^*, s_2^*) is a PSNE



Proof

(\implies) let (s_1^*, s_2^*) is a PSNE $\implies \overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax



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$$\Rightarrow u(s_1^*, s_2^*) \ge \max_{t_1 \in S_1} u(t_1, s_2^*)$$
$$\ge \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy}$$
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$$u(s_1^*, s_2^*) \ge \overline{v} \ge \underline{v} \ge u(s_1^*, s_2^*)$$
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Also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* respectively.



(\Leftarrow) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax $\implies (s_1^*, s_2^*)$ is a PSNE



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Similarly, we can show $u(s_1, s_2^*) \leq v$, $\forall s_1 \in S_1$ But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

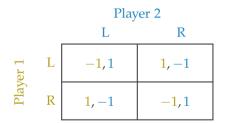
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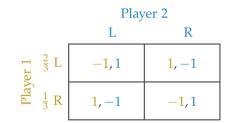
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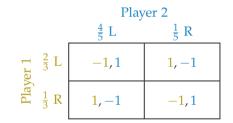




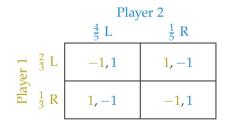








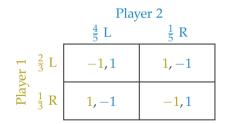




• Consider a finite set *A*, define

$$\Delta A = \{ p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1 \}$$





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• Mixed strategy set of player 1: $\Delta S_1 = \Delta \{L, R\}, (\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$



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- Utility of player *i* at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i,\sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \ u_i(s_1,s_2,\ldots,s_n)$$



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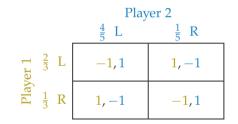


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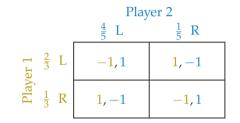
$$u_i(\sigma_i,\sigma_{-i}) = \sum_{s_1\in S_1}\sum_{s_2\in S_2}\cdots\sum_{s_n\in S_n}\sigma_1(s_1)\cdot\sigma_2(s_2)\cdots\sigma_n(s_n) \ u_i(s_1,s_2,\ldots,s_n)$$

- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.



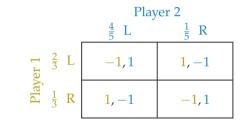






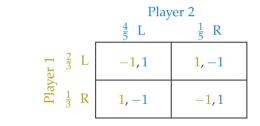
 $u_1(\sigma_1,\sigma_2)$





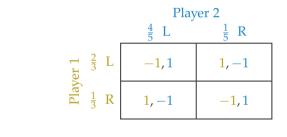
$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1)$$





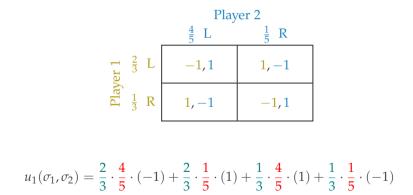
$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1)$$





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Definition (Mixed Strategy Nash Equilibrium)

A mixed strategy Nash equilibrium (MSNE) is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

 $u_i(\sigma_i^*, \sigma_{-i}^*) \ge u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$



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Question

Relation between **PSNE** and **MSNE**?



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| | Question |
|--|----------|
| Relation between PSNE and MSNE ? | |
| | |
| | Answer |
| $PSNE \implies MSNE$ | |



Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, is an **MSNE** if and only if $\forall s_i \in S_i$ and $\forall i \in N$

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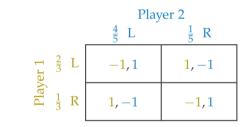
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$$\begin{split} u_i(\sigma_i, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &\leqslant \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(\sigma_i^*, \sigma_{-i}^*) \\ &= u_i(\sigma_i^*, \sigma_{-i}^*) \cdot \sum_{s_i \in S_i} \sigma_i(s_i) = u_i(\sigma_i^*, \sigma_{-i}^*) \end{split}$$

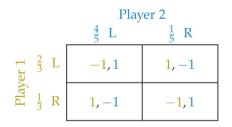


Is the mixed strategy profile an **MSNE**?



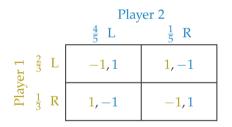
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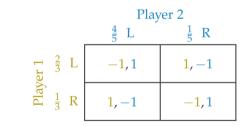




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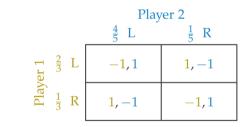


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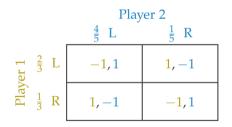
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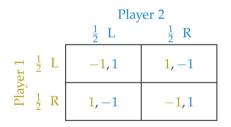
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- Some balance in the utilities is needed
- Does there exist any improving mixed strategy?

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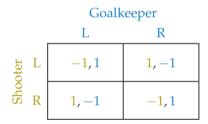


Consider Penalty Shoot Game

| | Goalkeeper | |
|-----------|-----------------------|-------|
| | L | R |
| oter T | -1,1 | 1, -1 |
| oys R | 1 , - 1 | -1,1 |



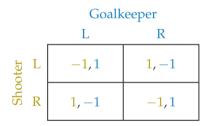
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Consider Penalty Shoot Game



Case 1: support profile ({*L*}, {*L*}): for player 1, $s'_1 = R$ – violates condition 2

Case 2: support profile $({L, R}, {L})$ – symmetric for the other case

For Player 1, the expected utility has to be the same for L and R - not possible – violates condition 1





Case 3: support profile $({L, R}, {L, R})$: condition 2 is vacuously satisfied



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$$u_1(L,(q,1-q)) = u_1(R,(q,1-q)) \implies (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \implies q = \frac{1}{2}$$



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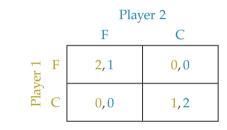
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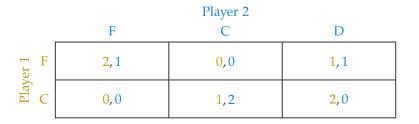
MSNE =

$$\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

Exercises







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MSNE Characterization Theorem



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A mixed strategy profile is an MSNE iff $\forall i \in N$

- $u_i(s_i, \sigma_{-i}^*)$ is identical $\forall s_i \in \delta(\sigma_i^*)$,
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Observations:

• $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$ maximizing w.r.t. a distribution \Leftrightarrow whole probability mass at max



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- If $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

the maximizer must lie in $\delta(\sigma_i^*)$ – if not, then put all probability mass on that $s'_i \notin \delta(\sigma_i^*)$ that has the maximum value of the utility – $(\sigma_i^*, \sigma_{-i}^*)$ is not a MSNE



 (\Rightarrow) Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_{i}(\sigma_{i}^{*},\sigma_{-i}^{*}) = \max_{\sigma_{i}\in\Delta S_{i}} u_{i}(\sigma_{i},\sigma_{-i}^{*}) = \max_{s_{i}\in S_{i}} u_{i}(s_{i},\sigma_{-i}^{*}) = \max_{s_{i}\in\delta(\sigma_{i}^{*})} u_{i}(s_{i},\sigma_{-i}^{*})$$
(1)



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(2)

Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: proves condition 1





We can shift the probability mass $\sigma^*(s_i)$ to s'_i , this new mixed strategy gives a strict higher utility to player *i*: contradicts MSNE



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Let
$$u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*)$$
 condition 1
Note $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$ condition 2



$$u_{i}(\sigma_{i}^{*},\sigma_{-i}^{*}) = \sum_{s_{i} \in \delta(\sigma_{i}^{*})} \sigma_{i}^{*}(s_{i})u_{i}(s_{i},\sigma_{-i}^{*}),$$



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$$= m_{i}(\sigma_{-i}^{*}) \qquad \text{previous conclusion}$$
$$= \max_{s_{i} \in S_{i}} u_{i}(s_{i}, \sigma_{-i}^{*}) \qquad \text{previous conclusion}$$
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$$\geq u_{i}(\sigma_{i}, \sigma_{-i}^{*}), \forall \sigma_{i} \in \Delta S_{i}$$

This proves the sufficient direction. The result yields an algorithmic way to find MSNE

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For every support profile $X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

Program

$$\begin{split} w_i &= \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N \\ w_i &\geq \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N \\ \sigma_j(s_j) &\geq 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_j} \sigma_j(s_j) = 1, \forall j \in N \end{split}$$

Remarks on the algorithm



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• This is not a linear program unless n = 2

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- For general game, there is no poly-time algorithm

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Remarks on the algorithm



Program

$$\begin{split} w_i &= \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N \\ w_i &\geq \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N \\ \tau_j(s_j) &\geq 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_j} \sigma_j(s_j) = 1, \forall j \in N \end{split}$$

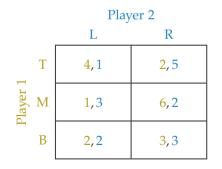
- This is not a linear program unless n = 2
- For general game, there is no poly-time algorithm
- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs]¹

¹Daskalakis, Goldberg, Papadimitriou, "The Complexity of Computing a Nash Equilibrium" [2009]



The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Is there a dominated strategy in this game? Domination can be via mixed strategies too





Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

Contents



Matrix games

- ▶ Relation between **maxmin** and PSNE
- ► Mixed Strategies
- Mixed Strategy Nash Equilibrium
- ► Find MSNE
- MSNE Characterization Theorem Proof
- ► Algorithm to find MSNE
- ► Existence of MSNE



Definition (Finite Games)

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Proof requires a few tools and a result from real analysis. Proof is separately given in the course webpage.



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- A set *S* ⊆ ℝⁿ is closed if it contains all its limit points (points whose every neighborhood contains a point in *S*). Example of a set that is not closed: [0, 1) every ball of radius *ε* > 0 around 1 has a member of [0, 1), but 1 is not in the set [0, 1).



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A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T : S \to S$, is **continuous** then *T* has a fixed point, i.e., $\exists x^* \in S$ s.t. $T(x^*) = x^*$.



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