CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 1: One sided matching problem

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

1.1 Two sided matching

Two-sided matching is a way to match two groups of agents where each group has preferences over who they want to match with. The goal is to create a stable and fair matching.
A matching is stable if there's no pair of agents (one from each side) who would prefer to be matched with each other instead of their current matches.

1.2 Fair division

Fair division is the process of dividing a resource among individuals in a way that is perceived as fair by all participants.

Heterogeneous goods are items that are not identical and may hold different levels of value or importance for different people. Examples include:

- Dividing land with different features (e.g., fertile vs. rocky areas).
- Allocating artwork, where tastes and preferences vary.

Normalization : $\forall i, v_i([0,1]) = 1$.

Proportional Division : If there are n agents, $\forall i, v_i(A_i) \geq 1/n$.

Envy-free division Envy-free division ensures that each participant in a resource division believes they have received a share that is at least as good as anyone else's share, based on their preferences. In other words, no one envies another participant's allocation.

$\forall i \forall j \quad v_i(A_i) \geq v_i(A_j)$

1.3 One sided matching

One-sided matching refers to the process of assigning items (e.g., goods, tasks, or slots) to agents when only the agents have preferences over the items, and the items do not have preferences over the agents. For example, house allocation. Here, money is not allowed and the only option is to exchange or reallocate the goods.

1.4 Formal Model

Set of objects $M = \{a_1, a_2, \ldots, a_m\}$

Set of agents $N = \{1, 2, \ldots, n\}, \quad m \geq n$

Each agent has a linear order over the objects. Types of linear orders:

- **Complete :** $\forall a, b \in M \quad$ either aR_ib or bR_ia
- **Transitive :** $aR_ib \quad$ and $\quad bR_ic \Rightarrow aR_ic$
- **Antisymmetric :** $aR_ib \quad$ and $\quad bR_ia \Rightarrow a = b$

P denotes the preference order for agent i:

- $P = \{P_1, P_2, ...P_n\}$ is a preference profile.
- $\mu =$ Set of all possible linear orders over M. $|\mu| = m!$
- $P_i(k, S)$: The k^{th} top object in $S \subseteq M$ in P_i.

AIM : Find a collective decision problem satisfying properties such as strategy proofness, Pareto efficiency, etc.

Definition 1.1 (Strategy proofness) *An SCF* $f : U^n \rightarrow A$ *is* ***Strategy Proof (SP)*** *if*
$f_i(P_i, P_{-i})\ P_i\ f_i(P_i^{'}, P_{-i}) \quad$ ***OR*** $\quad f_i(P_i, P_{-i}) = f_i(P_i^{'}, P_{-i}) \quad \forall P_i, P_{-i}, P_i^{'}\ \forall i \in N$

Definition 1.2 (Pareto Efficiency) *An SCF* f *is* ***Pareto Efficient (PE)*** *if for every possible profile* P*, there does not exist any other matching* $a \in A$ *and* $a \neq f(P)$ *such that* $a_i = f_i(P)$ *or* $a_i\ P_i\ f_i(P)\ \ \forall i \in N$.

1.4.1 Using the GS Theorem setting

By the *Gibbard-Satterthwaite (GS) Theorem*, we know that a Social Choice Function (SCF) f is $SP + ONTO \Leftrightarrow SP + PE$ iff f is dictatorial.

But the GS theorem doesn't hold here! WHY?
Here, the preferences are over objects and not over alternatives. In this context, an **alternative** is the matching/assignment of objects to agents. This matching or a mapping is injective (one-to-one) as each agent gets exactly one item and every item is allocated to at most one agent.
Hence, Set of alteratives $A = \{a : N \rightarrow M : injective\}$
Considering 2 alternatives a and a' where agent i gets the same item n both. This agent is always indifferent between these 2 alternatives. Hence, the domain of single object allocation violates the unrestricted assumption of the GS theorem.

1.4.2 Serial dictatorship mechanism

Let's consider a simple truthful mechanism. SCF $f : U^n \rightarrow A$. Define a fixed priority (serial dictatorship) mechanism.

Priority: an ordering over the agents; $\sigma : N \to N, bijection.$

Every agent in the order σ picks her favourite from the leftover list.

$$\begin{aligned} a(\sigma_1) &= P_{\sigma_1}(1, M) \\ a(\sigma_2) &= P_{\sigma_2}(1, M \backslash \{a(\sigma_1)\}) \\ &\vdots \\ f^\sigma(P) &= a \end{aligned}$$

Serial dictatorship is efficient.
Proof: Suppose f^σ is not efficient. Then $\exists P$ s.t. $\exists a \neq f^\sigma(P)$ satisfying $a(i)\ P_i\ f_i^\sigma(P)$ or $a(i) = f_i^\sigma(P)$, i.e., either agent i gets the same good or gets a better good, $\forall i \in N$.

Say, the first agent j in the priority order of σ that has $a(j)\ P_j\ f_j^\sigma(P)$. But it means that $a(j)$ was available when j's turn came to pick. According to the serial dictatorship, then agent j can't pick $f_j^\sigma(P)$ which is less preferred than $a(j)$. ∎

Serial dictatorship is strategy proof.
Proof: The agents before i in order σ pick goods and i cannot influence it. Agent i gets the best one from the remaining goods, therefore there is no reason to misreport.

∎

1.4.3 Modified serial dictatorship mechanism

$$N = \{1, 2, 3\}, \quad M = \{h_1, h_2, h_3\}$$

Here, h_i are the houses (goods).
The SCF is the same as serial dictatorship, but the priority order changes based on the preference of agent 1 in the following way:

$$\sigma = \begin{cases} (1, 2, 3), & \text{if } P_1(1) = h_1, \\ (2, 1, 3), & \text{if } P_1(1) \neq h_1. \end{cases}$$

Modified serial dictatorship is strategy proof.
Proof: Players 2 and 3 can't change the priority order; hence they can't manipulate. Player 1 can change it, but:

- **Case 1:** If $P_1(1) = h_1$, they get their most favorite house.
- **Case 2:** If $P_1(1) \neq h_1$, if they misreport to $P_1(1) = h_1$, they will be assigned h_1, which they prefer less than their top choice, say $b \in \{h_2, h_3\}$.

Since agent 1 picks second if they report truthfully, they can either get h_1 or better (if agent 2's preference is such that $P_2(1) \neq b$. Agent 1 can get h_1 only if $P_2(1) = b$ and $P_2(2) = h_1$). ∎

Modified serial dictatorship is efficient.
Proof: Every fixed priority serial dictatorship is efficient. The same argument as before applies. ∎