CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 10: EF1 and World of 'Bad's

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10.1 EF1 and Pareto Optimality

We have already observed that the notion of Envy Freeness (EF) is not guaranteed to exist for indivisible item allocations of objects. However, we arrived at a new notion of Envy Freeness upto 1 Good (EF1) which we proved exists for all valuation profiles over agents while allocating all objects.
An allocation A is Pareto Optimal $\iff$ there does not exist any allocation A' such that

$$\forall i \in N \ v_i(A'_i) \geq v_i(A_i)$$

$$\exists j \in N \ v_j(A'_j) > v_j(A_j)$$

Unfortunately, we are not guaranteed to be able to satisfy EF1 along with Pareto Optimality (PO) for indivisible item allocations in the general case. To summarize.

- EF: Does not always exist
- EF-1: Always exists for additive (Round robin) and monotone (ECE) valuations.

10.2 Nash Social Welfare

The idea of Nash Social Welfare (NSW) comes from cooperative games. The idea is to maximize the *product* of the net utility of all the agents.

Allocation $A = (A_1, ...A_n)$

$$NSW(A) = [v_1(A_1) \times v_2(A_2)...v_n(A_n)]^{\frac{1}{n}}$$

$$\text{optimal } NSW = \arg\max_A NSW(A)$$

If the optimal is 0, find the largest sets of agents who can simultaneously be given positive values and maximize the geometric mean wrt those agents.

How to satisfy both EF-1 and PO?

Theorem 10.1 *Any Nash optimal allocation satisfies EF-1 and PO [CK16]*

Proof: PO: If not PO, not NSW maximizing. Hence proved.

EF-1: Suppose allocation A is not EF-1. We show that it can't be NSW

Not EF-1 $\implies \exists i, j$ s.t. $\forall n_j \in A_j v_i(A_i) < v_i(A_j\{x_j\})$

Consider the ratio $\frac{v_j(\{g\})}{v_i(\{g\})}$

$$\arg \min_{g \in A_j, v_i(g) > 0} \frac{v_j(\{g\})}{v_i(\{g\})} = g^*$$

g^* is guaranteed to exist otherwise $v_i(A_j) = 0$ and i doesn't envy j. **Claim**: Transferring g^* from A_j to A_i improves NSW.

$$\text{To prove: } [v_i(A_i + v_i(g^*)][v_j(A_j) - v_j(g^*)] > v_i(A_i)v_j(A_j)$$

$$\implies v_j(A_j) > \frac{v_j(g^*)}{v_i(g^*)}[v_i(A_i + v_i(g^*)]$$

$$\begin{aligned}\frac{v_j(g^*)}{v_i(g^*)} &\leq \frac{v_j(g)}{v_i(g)} \\ &\leq \frac{\sum_{g \in A_j} v_j(g^*)}{\sum_{g \in A_j} v_i(g^*)} = \frac{v_j(A_j)}{v_i(A_j)} \end{aligned} \tag{1}$$

$$\text{From EF-1, } v_i(A_i) < v_i(A_j) - v_i(g^*)$$

$$\begin{aligned}\text{From (1): } v_j(A_j) &\geq \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j) \\ &> \frac{v_j(g^*)}{v_i(g^*)}[v_i(A_i + v_i(g^*)]\end{aligned}$$

■

Theorem 10.2 *Maximising NSW is APX-hard. [LE17]*

(APX-hard means hard to approximate. Related to reduction from the vertex cover on 3-regular graph problem)

Theorem 10.3 *An $EF1 + PO$ allocation is possible to find in pseudo-polynomial time. [BK18]*

This algorithm depends on the value of v_i's and thus we say it is pseudo-polynomial time. It is polynomial time on bounded valuations and is a 0.69-approximation to NSW.

10.3 Bads/Chores Allocation

Now we shall take a look at cases where the objects to be allocated are not items, but chores/tasks and the valuation of agents for such items is negative. As shown before, Envy Freeness does not exist, instead we look for $EF1$. The modified definition of $EF1$ for chores involves removing one chore from the allocation of the player who envies as described below.

10.3.1 Difference between Goods and Bads

If $v_i(S) =$ value of any bundle, $S \subseteq M$,

$$v_i(S \cup \{g\}) \geq v_i(S)\ \forall S \subseteq M$$

$$v_i(S \cup \{c\}) \leq v_i(S)\ \forall S \subseteq M$$

where g and c are goods and chores respectively, **for agent** i. Equivalently,
Goods have monotone non decreasing valuations for each agent,

$$v_i(S) \geq v_i(T) \quad \text{if} \quad T \subseteq S$$

And Chores have monotone non increasing valuations for each agent,

$$v_i(S) \leq v_i(T) \quad \text{if} \quad T \subseteq S$$

10.3.2 Envy Freeness

An allocation $A = (A_1, A_2, \dots A_n)$ is $EF1$ if $\forall i, j \in N$, $\exists x_i \in A_i$, such that,

$$v_i(A_i \setminus \{x_i\}) \geq v_i(A_j)$$

Now we arrive at the question of finding algorithms that can achieve this notion of $EF1$ envy freeness.

Theorem 10.4 *For additive valuations, Round Robin allocation is envy free for chores as well.*

Proof: Let us consider a pair of players i and j and look at who picks first.
Case I- i picks before j
As i picks before j, in every round, it individually does not envy the pick of j for that round as i's pick for that round has to be strictly better off for i. In the last round, if items run out before j has to pick one, still $EF1$ holds as i can just drop the last item they picked from the set and as valuations are additive, no envy across rounds means there is no envy overall.
Case II- j picks before i
Similar to the previous proof of $EF1$ in Round RObin allocation, i does not envy the pick of j chosen in the next round. We choose to exclude the pick of i in the last round and as the first pick of j only decreases their valuation (there are only chores, no goods), the allocation is $EF1$. ■

Is Round Robin $EF1$ for monotone valuations as well? Round Robin fails, as can be seen from the following example:

	A	B	C	D	E	{A,X} $\forall X$	{C,E}	{A,X,Y} $\forall X, Y$
1	-1	-2	-3	-4	-5	-100	-100	-100
2	-1	-2	-3	-4	-5	-100	-100	-100

All the valuations not mentioned are additive.
Agent 1 picks A, C, and E. Agent 1 envies agent 2 even after dropping any one chore from their bundle. We must try out Envy Cycle Elimination.

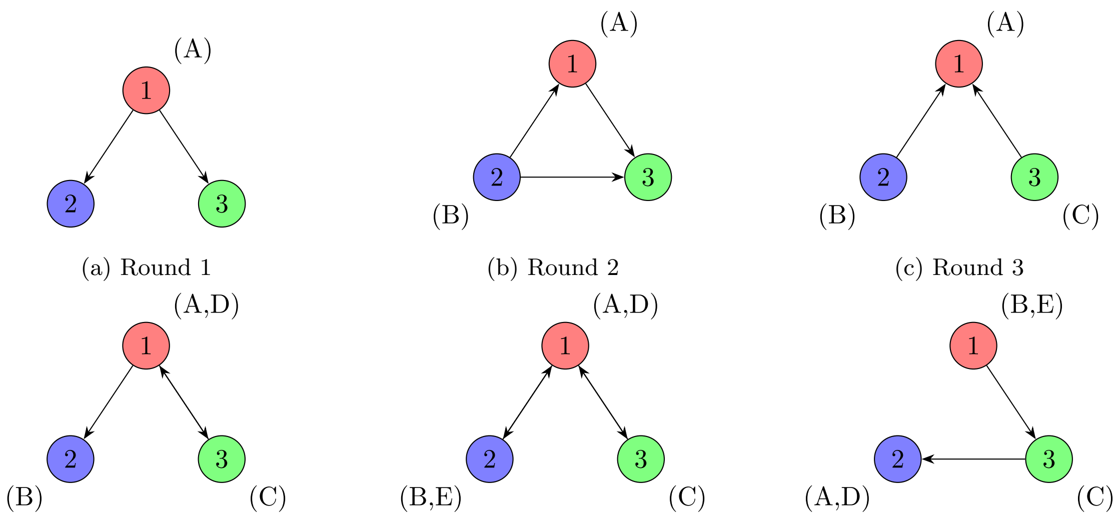
10.3.3 Envy Cycle Elimination

For ECE of Goods, we allocated items to source vertices. However, as Chores decrease valuation, it makes sense to allocate the chores to sink vertices.
Is traditional ECE guaranteed to be $EF1$ if we allocate chores to sink vertices? Unfortunately, no as we will see from the following example.

10.3.3.1 Example

Assume the valuations are additive for our convenience. Each edge in the graph denotes envy going from source to sink, bi-directional edges indicate both players envy each other.

Player	*A*	*B*	*C*	*D*	*E*	*F*
1	-1	-3	-2	-9	-3	.
2	0	-1	-4	-1	-2	.
3	0	-2	-1	0	-4	.



(a) Round 1 (b) Round 2 (c) Round 3

(d) Round 4 (e) Round 5 (f) Round 6

Now, as we had seen in the ECE algorithm previously, every step must satisfy $EF1$ for the allocated goods. However, our cycle resolution is Step 6 for allocating F breaks this rule as Player 1 envies Player 3 even if we remove any of their chores. The maximum valuation for Player 1 is $v_1(B) = v_1(E) = -3$ while the valuation for Player 3 with respect to Player 1 is $v_1(C) = -2$.
In the next lecture we shall see the modifications required to make ECE work for Chores as well.

References

[CK16] I. Caragiannis and D. Kurokawa et al., "The Unreasonable Fairness of Maximum Nash Welfare", 2016

[LE15] Euiwoong Lee, "APX-Hardness of Maximizing Nash Social Welfare with Indivisible Items", 2015

[BK18] S. Barman, S. Krishnamurthy and R. Vaish, "Finding Fair and Efficient Allocations", 2018