CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 12: Bargaining Games

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12.1 Introduction to Bargaining Games

Non-Cooperative Games: Games where agents are self-interested and cannot communicate among themselves before they decide their actions. This is what has been covered so far.

Cooperative Games: We now assume that agents can communicate before taking a decision. However, the agents are still self-interested. A type of such game is **Bargaining Game**.

Bargaining Game Setting: A set of possible outcomes are bargained on, and finally some outcome is recommended to the players by an arbitrator (trusted third party). In case the bargaining doesn't converge (i.e. the agents don't come to a common conclusion), there is a disagreement point which decides the outcome.

2-Player Bargaining Game Example

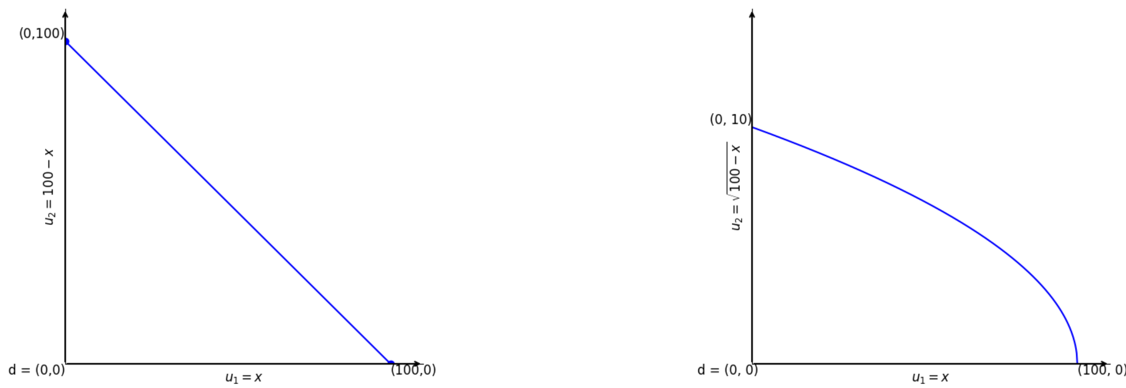
Assume we have distribute a reward of 100 between two agents. If the agents agree upon a distribution, then the money is distributed that way. Otherwise, no one gets any reward. The space of all *allocations* on which the agents can bargain is $S = \{(x, 100 - x) : x \in [0, 100]\}$ and the *disagreement point* is $d = (0, 0)$.

If the value for the money of the players is equal to the money itself, then $S = (x, 100-x)$ denotes the utility space of all possible allocations (see 12.1a). Here a reasonable division would be to split the money equally. If the utility of the players are $(x, \sqrt{100 - x})$ (i.e. the utility of the second player increases as a square root relation to the reward given), then the graph in this situation will correspond to 12.1b. In this case, the better decision could be to split the money such the the utilities of both the agents are equal.

12.2 Model

A Bargaining Game is an ordered pair (S, d), $S \subseteq \mathbb{R}^2$ and $d \in \mathbb{R}^2$:

- S is the set of alternatives (also called the allocation space). It is non-empty, compact, and convex set.
  - **Closed Set**: S is closed if all limit points of S lie in S. (A point p is a limit point of S if a ball of arbitrarily small radius around p contains at least one point from S that is distinct from p)
  - **Bounded**: S is bounded if $\exists (x_0 \in \mathbb{R}^n, R < \infty)$ such that a ball centered at x_0 with radius R contains S.
  - **Compact**: In $\mathbb{R}^n$, a set is compact if it is closed and bounded.



(a) Example 1

(b) Example 2

Figure 12.1: Introductory examples

  - **Convex**: $S \subset \mathbb{R}^n$ is convex if, for any two points $x, y \in S$, the entire line segment joining x and y lies completely within S. (i.e., $\forall t \in [0,1]$, $tx + (1-t)y \in S$)

- $d = (d_1, d_2)$ is the disagreement point.
- $\exists x \in S, x \gg d$. Note $\forall x, y \in \mathbb{R}^n$, $x \gg y \implies (\forall i \in \{1, 2, ..., n\}, x_i > y_i)$. This is to avoid the disagreement point being picked always. Additionally, we use the following notations to compare vectors $x, y \in \mathbb{R}^n$: (1) $x \geq y \implies (x_i \geq y_i, \forall i \in \{1, 2, ..., n\})$ and $(x_j > y_j,$ for some $j \in \{1, 2, .., n\})$; (2) $xy = (x_i y_i, \forall i \in \{1, 2, \ldots, n\})$.

Denote the collection of all bargaining games as $\mathcal{F}$. Given an instance $(S, d) \in \mathcal{F}$, a *solution concept* should find a point in S that satisfies a set of desirable properties. Formally, a solution concept $\phi : \mathcal{F} \to S$ is a function defined from a bargaining game to $\mathbb{R}^2$ i.e., $\phi(S, d) \in S, \forall (S, d) \in \mathcal{F}$.

12.3 Design desiderata

We first consider the *symmetry* property for the solution concept. A bargaining game (S, d) is symmetric if, $d_1 = d_2$ and $\forall (x_1, x_2) \in \mathbb{R}^2$, $(x_1, x_2) \in S \implies (x_2, x_1) \in S$.

Definition 12.1 (Symmetry (SYM)) *A solution concept ϕ is called symmetric if for all symmetric bargaining games (S, d), $\phi_1(S, d) = \phi_2(S, d)$.*

Next, we require the solution concept to provide an *efficient* alternative. An alternative $x \in S$ is an efficient point if $\nexists y \in S, y \neq x$ such that $y \geq x$. Similarly, an alternative $z \in S$ is a *weakly efficient* point if $\nexists y \in S, y \neq z$ such that $y \gg z$. Note that if a point is efficient, it is also weakly efficient. Denote PO(S) to be the set of all *efficient* points in S.

Definition 12.2 (Efficiency (PO/EFF)) *A solution concept ϕ is efficient if $\forall (S, d) \in \mathcal{F}, \quad \phi(S, d) \in PO(S)$.*

The next property we consider is *Covariance under Positive Affine Transformation (CPAT)*. The motivation for this property is that the solution concept should be scale free (independent of the units of the utility). Also, it should be affected the same way a translation affects all possible allocations i.e. $aS + b = \{(a_1 s_1 + b_1,\ a_2 s_2 + b_2) : (s_1, s_2) \in S\}$. Similarly, $ad + b = (a_1 d_1 + b_1, a_2 d_2 + b_2)$.

Definition 12.3 (Covariance under Positive Affine Transformation (CPAT)) *A solution concept ϕ is CPAT if $\forall (S, d) \in \mathcal{F}$, and for every $a \in \mathbb{R}^2, a \gg (0, 0),\ b \in \mathbb{R}^2,\ \phi(aS + b, ad + b) = a\phi(S, d) + b$.*

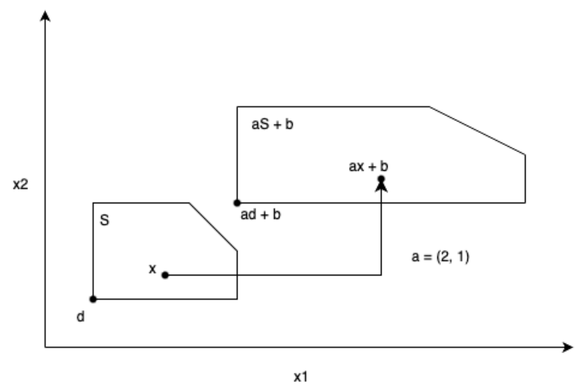


Figure 12.2: Covariance under Positive Affine Transformation

Definition 12.4 (Independence of Irrelevant Alternatives (IIA)) *A solution concept ϕ satisfies IIA if $\forall (T, d) \in \mathcal{F}$ and $S \subseteq T$ we have, $\phi(T, d) \in S \implies \phi(S, d) = \phi(T, d)$.*

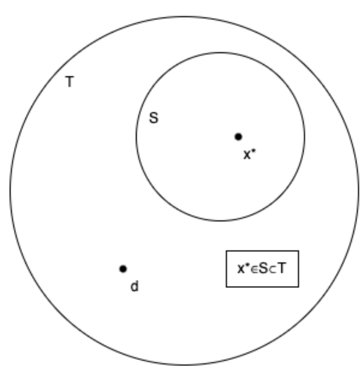


Figure 12.3: Independence of Irrelevant Alternatives

12.4 Nash Bargaining Solution

Theorem 12.5 *There exists a unique solution concept $\mathcal{N}$ for the family of bargaining games $\mathcal{F}$ which satisfies SYM, EFF, CPAT, and IIA, and is given by:*

$$\mathcal{N}(S,d) = \underset{\substack{(x_1,x_2)\in S\\ x\geq d}}{argmax}\ (x_1-d_1)\cdot(x_2-d_2)$$

i.e. we choose x such that the shaded area in 12.4 is maximized.

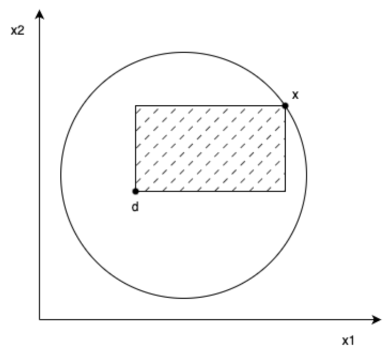


Figure 12.4: Nash Solution

To prove the above theorem, we need to prove three things:

- Nash solution is unique
- Nash solution satisfies the four required properties
- Any solution satisfying the four properties is the same as Nash solution

12.5 Nash Solution is Unique

Claim 12.6 *A unique $x \in S, x \geq d$ maximizes $(x_1-d_1)(x_2-d_2)$.*

Proof: As a first step, we work in the coordinate space $S-d$, such that the disagreement point becomes $(0,0)$.Therefore, the Nash Solution becomes

$$\underset{\substack{(z_1,z_2)\in S-d\\ z\geq(0,0)}}{\text{argmax}}\ z_1\cdot z_2.$$

Note that the value of the product is unchanged due to the coordinate transformation. In addition, the function is continuous and the domain on which it is maximized is convex, compact and non-empty (by assumption on S). Hence, an optimal solution always exists. Suppose that the solution is not unique, i.e. $\exists y,v \in (S-d),\ y,v \geq (0,0),\ y \neq v$ such that $y_1y_2 = v_1v_2 = c^*$, which is the optimal value. Assume WLOG that $y_1 > v_1, y_2 < v_2$ and construct $w = 0.5(y+v)$. Since S is convex, $w \in S$.

Next we define the following positive values (refer 12.5 for clarity).

- $v_1 y_2 = A$
- $y_2(y_1 - v_1) = B$
- $v_1(v_2 - y_2) = D$
- $(y_1 - v_1)(v_2 - y_2) = C$

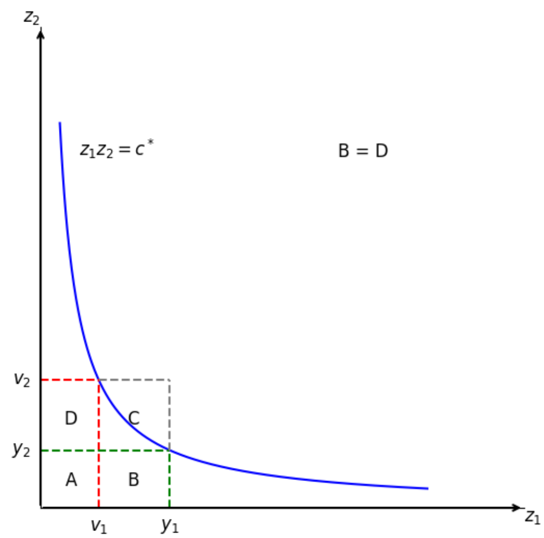


Figure 12.5: Uniqueness of Nash solution

Now,

$$
\begin{aligned}
w_1 w_2 &= (\frac{y_1 + v_1}{2})(\frac{y_2 + v_2}{2}) \\
&= \frac{y_1 y_2 + v_1 v_2 + y_1 v_2 + y_2 v_1}{4} \\
&= \frac{(A + B) + (A + D) + (A + B + C + D) + (A)}{4} \\
&= \frac{2(A + B + A + D) + C}{4} \\
&= \frac{2(c^* + c^*) + C}{4} = c^* + \frac{C}{4} > c^* = y_1 y_2 = v_1 v_2
\end{aligned}
$$

This implies neither y nor v is the Nash solution, which leads to a contradiction. Therefore, the Nash Solution is unique. ■