भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

Strategic Multi-agent Artificial Intelligence

Day 1: AI and competitive games

Swaprava Nath

Slide preparation acknowledgments: Drashthi Doshi

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

▶ Part 1

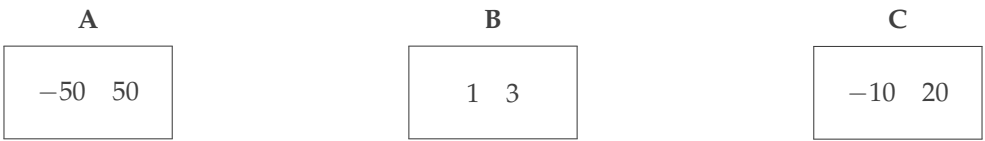
- Sequential move games
- Partial information games
- Subgame perfection

▶ Part 2

- Solutions for 2 player zero sum games
- Simultaneous move games

A 2-player game

Consider a two-player game in which there are 3 buckets (A, B, C) and each bucket contains two numbers as shown below.



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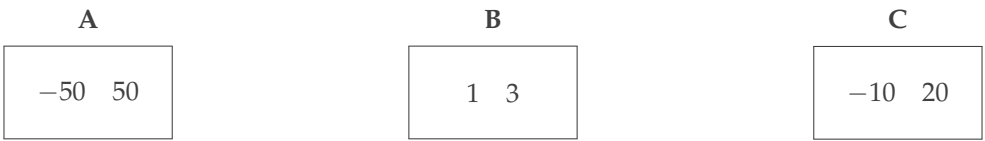

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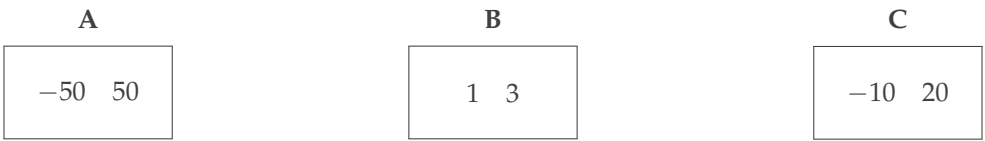

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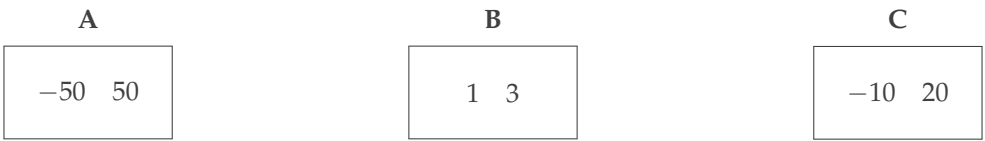

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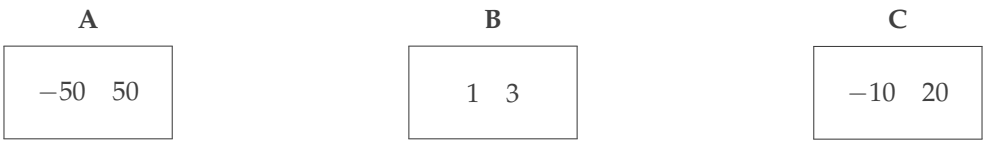

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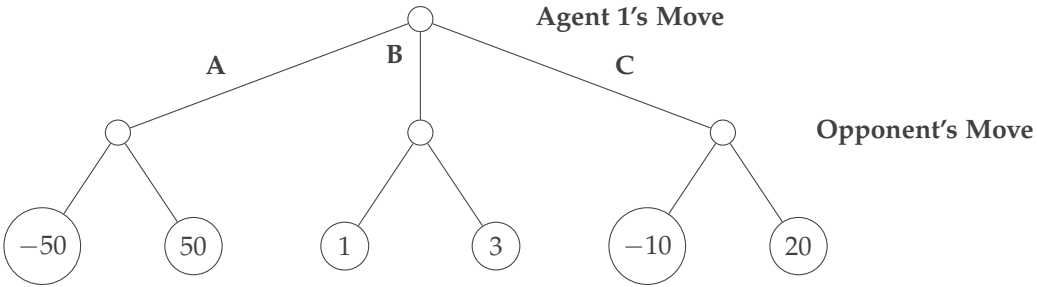

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So in this case agent picks **bucket C**, to maximize it's utility.

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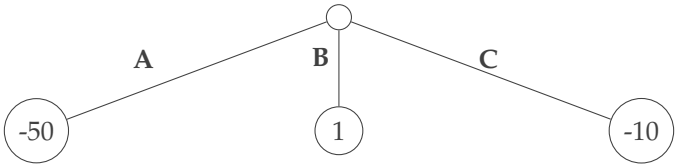

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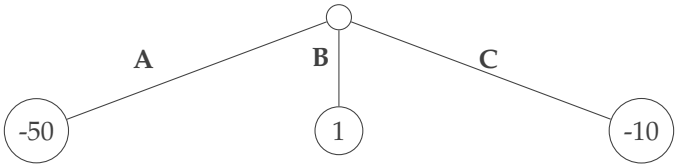
In this case,the agent picks **Bucket B**, as $\max\{-50,\ 1, -10\} = 1$.



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ii) Utility(s) is not defined in other (intermediate) states.

2-player zero sum games

For example in the game of chess

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- **isEnd(s):** Whether 's' is a checkmate or a draw
- Utility function is defined as

$$\textbf{Utility}(s) = \begin{cases} +M & \text{if white wins} \\ -M & \text{if black wins/white loses} \\ 0 & \text{if it is a draw} \end{cases}$$

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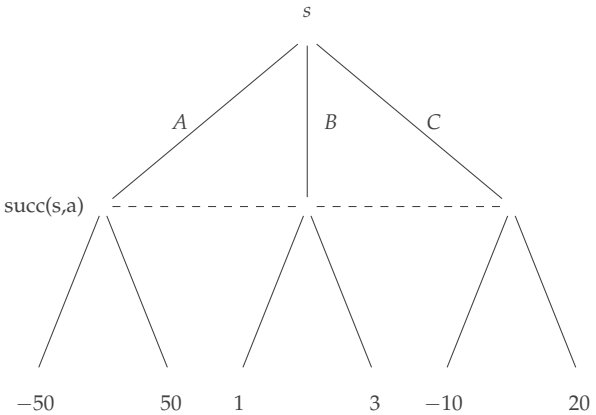
Games with partial information

Consider the game tree below:



$a \in \{A, B, C\}$

Games with partial information

$$u_{agent}(s) = \begin{cases} \text{utility(s)} & \textit{if } \text{isEnd(s)} \\ \sum_{a \in actions(s)} \Pi_{agent}(s)[a] \ \text{u}_{agent}(\text{successor}(s,a)) & \textit{if } \text{player(s) = agent} \\ \sum_{a \in actions(s)} \Pi_{opponent}(s)[a] \ \text{u}_{agent}(\text{successor}(s,a)) & \textit{if } \text{player(s) = opponent} \end{cases}$$

`Note:` $\Pi_{agent}(s)[a]$ and $\Pi_{opponent}(s)[a]$ are probabilities that the agent and the opponent pick action 'a' respectively.

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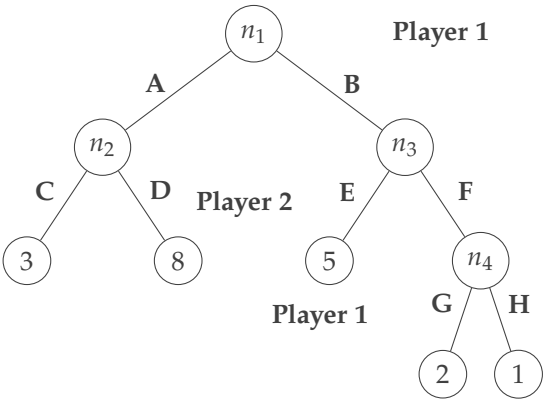

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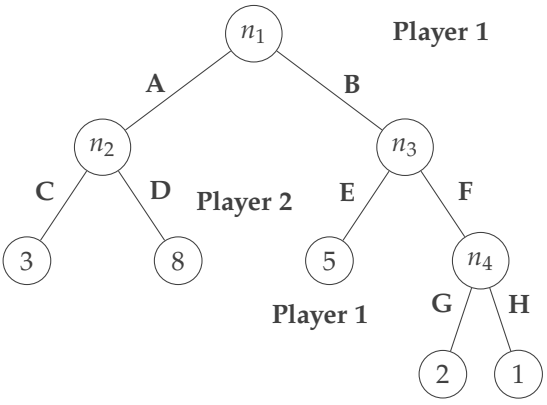

- **Player 1** is a utility maximizer (**Max Player**) i.e. plays to maximize his utility.
  Hence if player(s) = player 1, the utility of Player 1 changes to,
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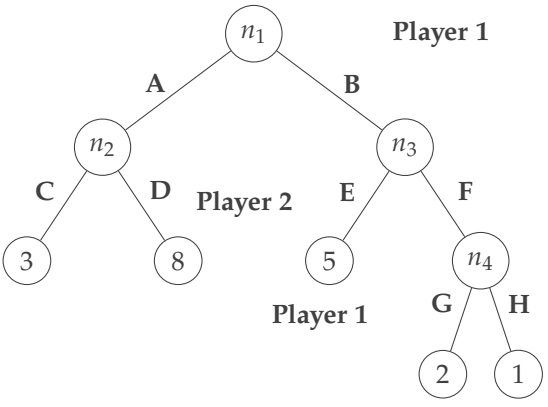

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- **Player 2** is the utility minimiser(**Min Player**) i.e. plays to minimize his utility.
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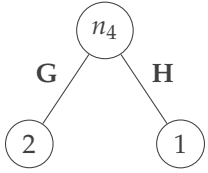

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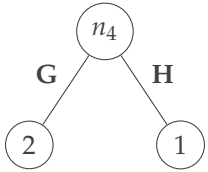

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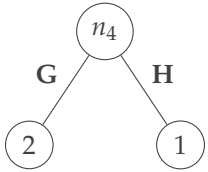

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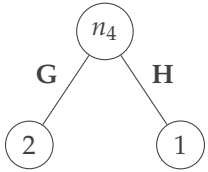

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We are done solving this subgame.

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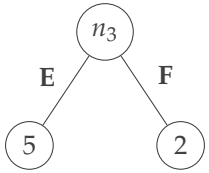

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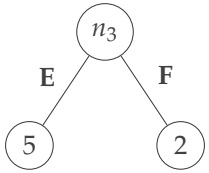

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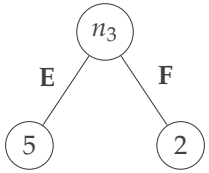

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Therefore, he should pick **F** so that player 1 gets **2** (from n_4, instead of getting **5** by picking **E**).

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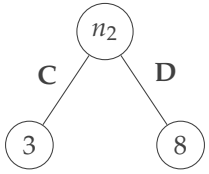

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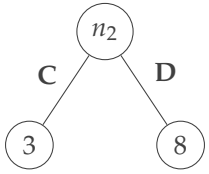

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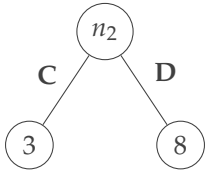

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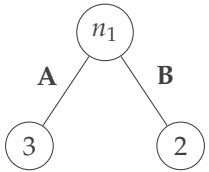

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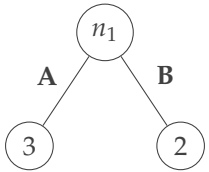

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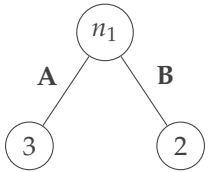

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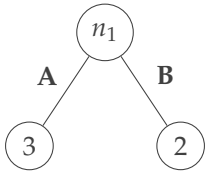

In this subgame, it's Player 1's turn.

Player 1 (max player) should pick A here to get **3** (> 2, that he would have got by choosing **B**).

Therefore, the **final utility of the Player 1 = 3**

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The algorithm described in the above example is **Backward Induction**.

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Checkers was solved in 2007 after 18 years of computation and the optimal solution was a **draw**.

Contents

▶ Part 1

• Sequential move games • Partial information games • Subgame perfection

▶ Part 2

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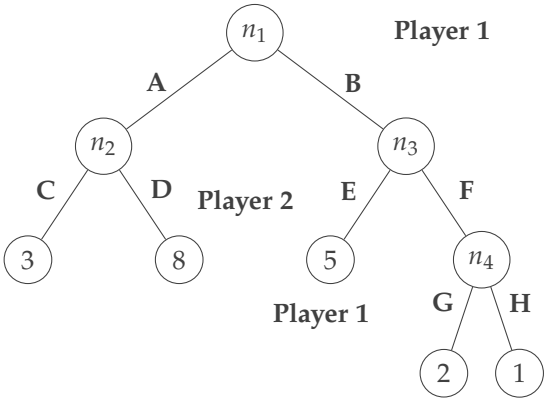
Recall: 2-player zero sum game



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The utility of the agent at node s and depth d can be written as

$$u_{agent}(s,d) = \begin{cases} utility(s) & \text{if } isEnd(s) \\ eval(s) & \text{if } d = 0 \\ \max\limits_{a \in actions(s)} \{u_{agent}(succ(s,a), d-1)\} & \text{if } player(s) = agent \\ \min\limits_{a \in actions(s)} \{u_{agent}(succ(s,a), d-1)\} & \text{if } player(s) = opponent \end{cases}$$

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Here K, Q, R and K', Q', R' are the number of king, queen and rooks that the agent and the opponent respectively have.

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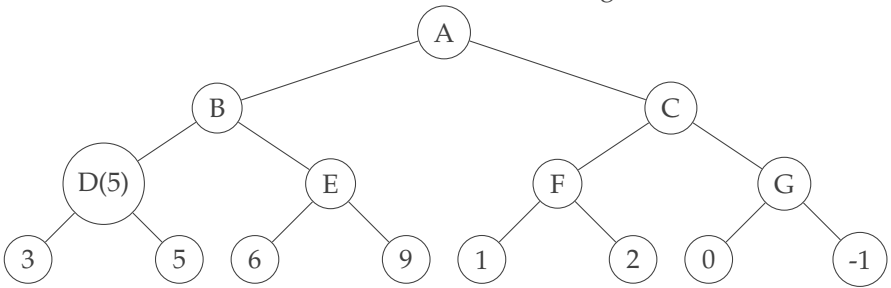
- — For 'Max' nodes: $\alpha = \max(\alpha, \text{value})$
  — For 'Min' nodes: $\beta = \min(\beta, \text{value})$

An example of α, β pruning

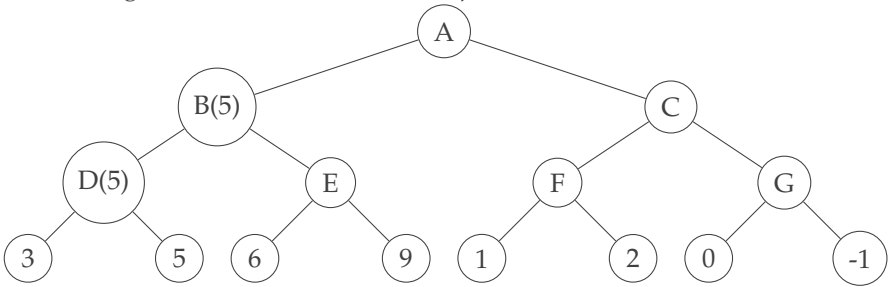
Considering the given binary tree, these are the following steps :-

1. Start with the subtree rooted at D. After transfering values from its two leaves, α=5, β=∞.



An example of α, β pruning

1. Transfering the value of D to B, α=-∞, β=5.



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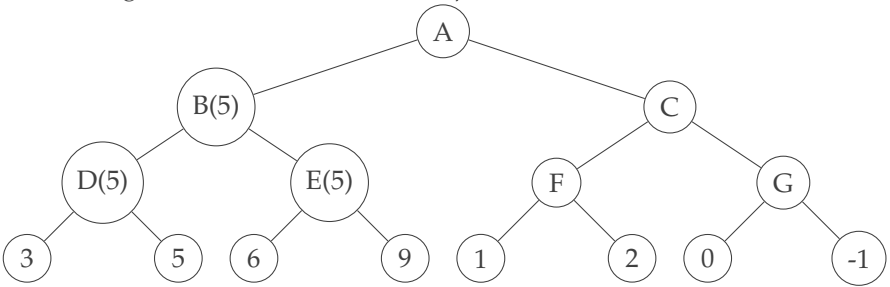
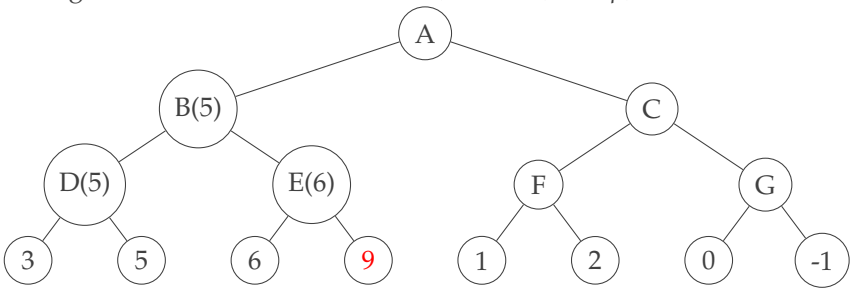
1. Transfering the value of B to E, α=-∞, β=5.



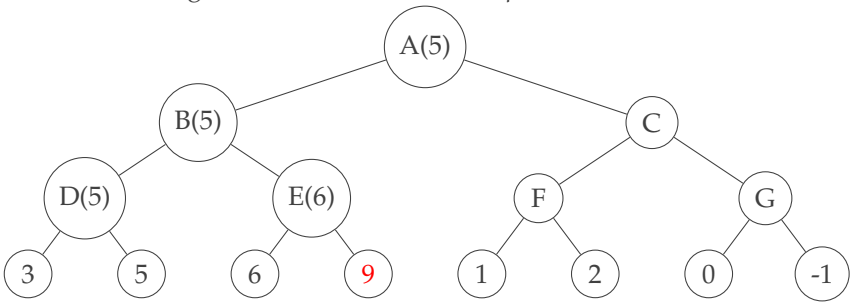
An example of α, β pruning

1. Taking the value from first child of E, α=6. So, $\alpha > \beta$, hence the rest branch of E is ignored.



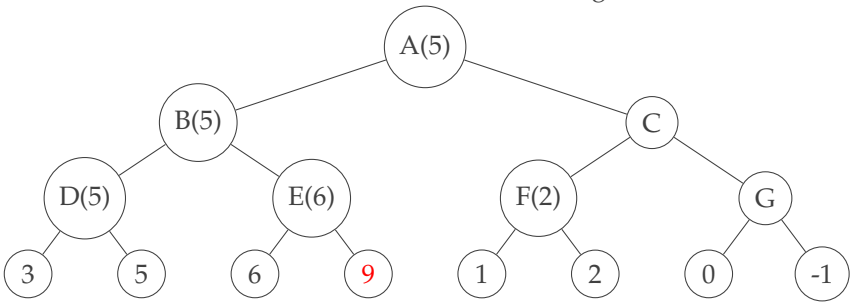
An example of α, β pruning

1. After transfering values from B to A, α=5, β=∞.



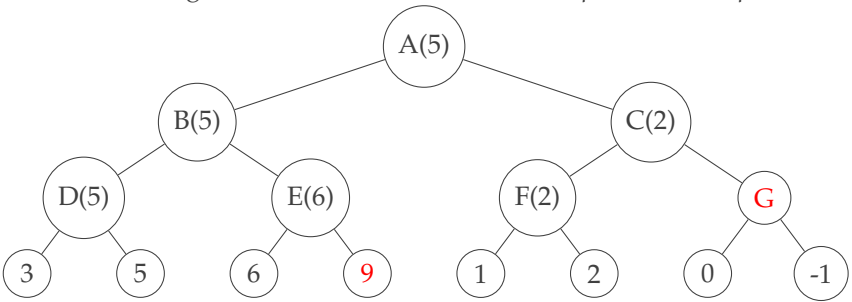
An example of α, β pruning

1. Start with the subtree rooted at F. After transfering values from its two leaves, α=2, β=∞.



An example of α, β pruning

- After transfering values from A and F to C, α=5, β=2. So, $\alpha > \beta$. Hence, G subtree is ignored.



Likewise, the whole game can be solved in a computionally less expensive way by pruning.

Contents

▶ Part 1

• Sequential move games • Partial information games • Subgame perfection

▶ Part 2

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This is an example of a **2-player zero sum simultaneous move game** and is called a **Matrix Game**.

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Definition

Equilibrium : A tuple of actions from which no player **gains** (a strict gain) by a **unilateral deviation**. A unilateral deviation from a tuple of actions $(a_1, a_2, \ldots, a_n)$ is a tuple of actions of the form $(b_1, b_2, \ldots, b_n)$ where $a_i = b_i$, $\forall i \neq j$ for some j ie, the actions of exactly one player j differs from the previous set.

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Therefore there exists no such equilibrium for the game.

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In this case , (L, L) is a simultaneous move equilibrium for game 2 because, the agent (player 1) will not gain any extra utility from (C, L) or (R, L) and similarly the opponent (player 2) will not gain any extra utility from (L, C) or (L, R).

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If for some game, these two quantities equal each other, we have an equilibrium and the equilibrium point is called the saddle point.

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Since the previous inequality is true $\forall s_1$, it is also true for s_1^* which maximises $\min_{s_2} u(s_1, s_2)$.

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Similarly, since the previous inequality is true $\forall s_2$, it is also true for s_2^* which minimises $\max_{s_1} u(s_1, s_2)$. Therefore we have,

$$\max_{s_1} \min_{s_2} u(s_1, s_2) \leqslant \max_{s_1} u(s_1, s_2^*)$$
$$\Rightarrow \max_{s_1} \min_{s_2} u(s_1, s_2) \leqslant \min_{s_2} \max_{s_1} u(s_1, s_2)$$

And we are done with the proof. □

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